Dirichlet Forms

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These notes are based on parts of Sections 1.1-1.4 and 2.1-2.2 of Fukushima, Oshima, and Takeda.

1 Dirichlet forms – basic notions

Let H be a Hilbert space over the reals. We have a symmetric form $\mathcal{E}$ defined on $D(\mathcal{E}) \times D(\mathcal{E})$ where $D(\mathcal{E})$ is dense in H. $\mathcal{E}$ satisfies the following.

(1) $\mathcal{E}(u,v) = \mathcal{E}(v,u)$.

(2) $\mathcal{E}(u+v,w) = \mathcal{E}(u,w) + \mathcal{E}(v,w)$.

(3) $a\mathcal{E}(u,v) = \mathcal{E}(au,v)$.

(4) $\mathcal{E}(u,u) \geq 0$.

We let (u,v) be the usual inner product in H. Define

$$\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v), \qquad u,v \in D(\mathcal{E}).$$

Set $D(\mathcal{E}_\alpha) = D(\mathcal{E})$.

$\mathcal{E}_\alpha, \mathcal{E}_\beta$ generate equivalent metrics when $\alpha \neq \beta$. To see this,

$$(u,u) \leq \frac{1}{\alpha}\mathcal{E}_\alpha(u,u),$$

so

$$\mathcal{E}_\beta(u,u) = \mathcal{E}(u,u) + \beta(u,u) \leq (1 + (\beta/\alpha))\mathcal{E}_\alpha(u,u).$$

$\mathcal{E}$ is closed in $D(\mathcal{E})$ means that $D(\mathcal{E})$ is complete with respect to the measure generated by $\mathcal{E}_\alpha$. This means that if $u_n \in D(\mathcal{E})$ and $\mathcal{E}_1(u_n - u_m, u_n - u_m) \to 0$, then there exists $u \in D(\mathcal{E})$ such that $\mathcal{E}_1(u_n - u, u_n - u) \to 0$.

$\mathcal{E}$ is closable if $\mathcal{E}(u_n - u_m, u_n - u_m) \to 0$ and $(u_n, u_n) \to 0$, then $\mathcal{E}(u_n, u_n) \to 0$.

$\mathcal{E}^2$ is an extension of $\mathcal{E}^1$ if $D(\mathcal{E}^1) \subset D(\mathcal{E}^2)$ and $\mathcal{E}^2 = \mathcal{E}^1$ on $D(\mathcal{E}^1) \times D(\mathcal{E}^1)$.

Proposition 1.1 *A symmetric form $\mathcal{E}$ has a closed extension if and only if $\mathcal{E}$ is closable.*

Proof. Suppose $\mathcal{E}$ is closable. Let $\mathcal{A}$ be the set of all $\mathcal{E}_1$-Cauchy sequences. Say u_n is equivalent to u_n' if $\mathcal{E}_1(u_n - u_n', u_n - u_n') \to 0$. Let $D(\overline{\mathcal{E}})$ be the class of all equivalent classes. $\overline{\mathcal{E}}$ is a closed symmetric form on H which is a closed extension of $\mathcal{E}$. □

Let $H = L^2(X; m)$, so that $(u, v) = \int_X u(x)v(x)\, m(dx)$.

$\mathcal{E}$ is Markovian if for all $\varepsilon > 0$ there exists $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that

(1)
$$\phi_\varepsilon(t) = t, \qquad t \in [0, 1];$$

(2) $-\varepsilon \le \phi_\varepsilon(t) \le 1 + \varepsilon$;

(3) $0 \le \phi_\varepsilon(t') - \phi_\varepsilon(t) \le t' - t$; and

(4) if $u \in D(\mathcal{E})$, then $\phi_\varepsilon(u) \in D(\mathcal{E})$ and
$$\mathcal{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \le \mathcal{E}(u, u).$$

Saying unit contractions operate on $\mathcal{E}$ means that if $u \in D(\mathcal{E})$ and $v = (0 \vee u) \wedge 1$, then $v \in D(\mathcal{E})$ and $\mathcal{E}(v, v) \le \mathcal{E}(u, u)$.

Saying normal contractions operate on $\mathcal{E}$ means the following. A normal contraction v of u is a function such that $|v(x)| \le |u(x)|$ for all x and $|v(x) - v(y)| \le |u(x) - u(y)|$ for all x, y. Then if $u \in D(\mathcal{E})$ and v is a normal contraction of u, then $v \in D(\mathcal{E})$ and $\mathcal{E}(v, v) \le \mathcal{E}(u, u)$.

If normal contractions operate on $\mathcal{E}$, then unit contractions operate on $\mathcal{E}$. If unit contractions operate on $\mathcal{E}$, then $\mathcal{E}$ is Markovian. These three notions will turn out to be equivalent if $\mathcal{E}$ is closed.

We let C_b be the continuous bounded functions, C_0 the continuous functions with compact support, and C_∞ the continuous functions that vanish at ∞, which means that given ε there exists a compact set K depending on ε such that $|f| < \varepsilon$ outside of K.

$\mathcal{C}$ is a core of $\mathcal{E}$ if $\mathcal{C} \subset D(\mathcal{E}) \cap C_0$ such that $\mathcal{C}$ is dense in $D(\mathcal{E})$ with respect to $\mathcal{E}_1$ norm and dense in C_0 with respect to the sup norm. $\mathcal{E}$ is regular if $\mathcal{E}$ possesses a core.

$\mathcal{E}$ is local if $u, v \in D(\mathcal{E})$ and supp u, supp v disjoint sets implies $\mathcal{E}(u,v) = 0$. Here supp u is the support of the measure $u(x)\, m(dx)$. When u is continuous, this is the same as the usual support of a function.

2 Examples

Let $D \subset \mathbb{R}^d$ and $D(\mathcal{E}) = C_0^\infty(D)$, i.e., those functions whose support is contained in D. Suppose $a_{ij}(x)$ is the $(i-j)^{th}$ entry of the matrix $a(x)$ and $a(x)$ are matrices that are bounded above and are strictly elliptic, which means that they are positive definite for each x. Define

$$\mathcal{E}(u,v) = \sum_{i,j=1}^d \int_D \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x)\, a_{ij}(x)\, dx.$$

The symmetry is obvious. That $\mathcal{E}$ is local is easy. To show that it is Markovian, let ϕ_ε be a function that equals t for $0 \le t \le 1$, is $-\varepsilon$ for $t < -2\varepsilon$, is $1+\varepsilon$ for $t > \varepsilon$, and is smooth. Then

$$\mathcal{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) = \sum \int |\phi_\varepsilon'(u)|^2 \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x)\, a_{ij}(x)\, dx.$$

Since a_{ij} is positive definite, this is less than or equal to $\mathcal{E}(u,u)$.

The form is not closed because $C_0^\infty(D)$ is too small. If a_{ij} is uniformly positive definite, we can extend $\mathcal{E}$ to $H_0^1(D) \times H_0^1(D)$, where $H_0^1(D)$ is the closure of $C_0^\infty(D)$ with respect to the norm $(\int |\nabla f|^2\, dx)^{1/2}$, and then $\mathcal{E}$ becomes closed.

For another example, let

$$\mathcal{E}(u,v) = \int \int (u(y) - u(x))(v(y) - v(x)) J(x,y)\, dy\, dx,$$

where J is a symmetric function satisfying some reasonable boundedness conditions. (E.g., that $\int (1 \wedge |y-x|^2) J(x,y)\, dy$ is a bounded function of x.)

3 Closed forms and semigroups

For semigroups, we want T_t to be a symmetric operator with domain H, $T_t T_s = T_{t+s}$, and we want T_t to be contractive, which means $(T_t u, T_t u) \le$

(u,u). (This is equivalent to $\|T_t\| \leq \|u\|$.) T_t is strongly continuous if $\|T_t u - u\| \to 0$ as $t \to 0$ for all u.

For resolvents, which we write $G_\alpha f$, we want G_α to be symmetric with domain H, to satisfy the resolvent equation

$$G_\alpha - G_\beta + (\alpha - \beta) G_\alpha G_\beta = 0,$$

and to be contractive:

$$(\alpha G_\alpha u, \alpha G_\alpha u) \leq (u,u).$$

G_α is strongly continuous if $\|\alpha G_\alpha u - u\| \to 0$ as $\alpha \to \infty$.

Let A be a negative self-adjoint operator (e.g., $Af = -f''$). By spectral theory, we write

$$Af = \int_0^\infty (-\lambda)\, E(d\lambda) f.$$

If

$$T_t f = \int_0^\infty e^{-\lambda t}\, E(d\lambda) f$$

and

$$G_\alpha f = \int_0^\infty \frac{1}{\lambda + \alpha}\, E(d\lambda) f,$$

we see that

$$\lim_{h \to 0} \frac{T_h f - f}{h} = Af, \qquad f \in D(A),$$

and

$$G_\alpha f = \int_0^\infty e^{-\alpha t} T_t f\, dt.$$

Also

$$(\alpha - A) G_\alpha f = G_\alpha (\alpha - A) f = f.$$

By the Hille-Yosida theorem

$$T_t f = \lim_{\beta \to \infty} e^{-t\beta} \sum_{n=0}^\infty \frac{(t\beta)^n}{n!} (\beta G_\beta)^n f,$$

which is just another way of saying

$$T_t f = \lim e^{t\beta(\beta G_\beta - I)} f.$$

Theorem 3.1 *There is a one-to-one correspondence between closed symmetric forms $\mathcal{E}$ and non-positive self adjoint operators on H given by*

$$D(\mathcal{E}) = D(\sqrt{-A})$$

and

$$\mathcal{E}(u,v) = (\sqrt{-A}u, \sqrt{-A}v).$$

Proof. (1) Suppose $-A$ is a positive operator and self-adjoint. Then so is $\sqrt{-A}$. Hence $\sqrt{-A}$ is closed.

$\mathcal{E}$ as defined above is closed: if $u_n \in D(\sqrt{-A})$ and $u_n \to u$, $\sqrt{-A}u_n \to w$, then $u \in D(\sqrt{-A})$ and $\sqrt{-A}u = w$.

Define G_α to be the resolvent for A. Then

$$G_\alpha(H) \subset D(\mathcal{E})$$

and

$$\mathcal{E}_\alpha(G_\alpha u, v) = (u,v), \qquad u \in H, v \in D(\mathcal{E}).$$

This follows from

$$D(\mathcal{E}) = \Big\{ u \in H : \int_0^\infty \lambda\,(E(d\lambda)u,u) < \infty$$

and

$$\mathcal{E}(u,v) = \int_0^\infty \lambda\,(E(d\lambda)u,v).$$

(2) Let $\mathcal{E}$ be a closed symmetric form on H. Define $\ell(v) = (u,v)$, and notice

$$|\ell(v)| \le \|u\|\,\|v\| \le \|u\|(\frac{1}{\alpha}\mathcal{E}_\alpha(v,v)).$$

So $\ell(v)$ is a bounded linear functional on the Hilbert space $D(\mathcal{E})$ with norm $(\mathcal{E}_\alpha(u,u))^{1/2}$. So there exists $G_\alpha u \in D(\mathcal{E})$ such that

$$\mathcal{E}_\alpha(G_\alpha u, v) = (u,v), \qquad v \in D(\mathcal{E}).$$

We leave it as an exercise to show that G_α is symmetric and that the resolvent equation holds for G_α.

G_α is contractive:

$$\alpha(G_\alpha u, G_\alpha u) \le \mathcal{E}_\alpha(G_\alpha u, G_\alpha u) = (u, G_\alpha u).$$

Now use Cauchy-Schwarz to get the contraction property.

Since βG_β is contractive and $D(\mathcal{E})$ is dense in H, to show strong continuity, it suffices to look at $u \in D(\mathcal{E})$.

$$\begin{aligned}\beta(\beta G_\beta u - u, \beta G_\beta u - u) &\le \mathcal{E}_\beta(\beta G_\beta u - u, \beta G_\beta u - u)\\ &= \beta^2(G_\beta u, u) - \beta(u,u) + \mathcal{E}(u,u)\\ &\le \mathcal{E}(u,u),\end{aligned}$$

so $\beta G_\beta u \to u$.

Let A be the generator of G_α. $-A$ is non-negative and self-adjoint, and let $\mathcal{E}'$ be the associated symmetric form. We claim $\mathcal{E} = \mathcal{E}'$. Note $G_\alpha(H) \subset D(\mathcal{E}')$, and

$$\mathcal{E}'_\alpha(G_\alpha u, G_\alpha v) = (u, G_\alpha v) = \mathcal{E}_\alpha(G_\alpha u, G_\alpha v).$$

So $\mathcal{E}' = \mathcal{E}$ on $G_\alpha(H) \times G_\alpha(H)$. Now use the fact that $G_\alpha(H)$ is dense in both $D(\mathcal{E}')$ and in $D(\mathcal{E})$.

Note that given $\mathcal{E}$, G_α is uniquely determined, so A is uniquely determined.

□

Corollary 3.2 *The correspondence is given by* $D(A) \subset D(\mathcal{E})$ *and*

$$\mathcal{E}(u,v) = (-Au, v), \qquad u \in D(A), v \in D(\mathcal{E}).$$

Lemma 3.3 *(1)* $T_t(H) \subset D(\mathcal{E})$.

(2)

$$\mathcal{E}(T_t u, T_t u) \le \frac{1}{2t}[(u,u) - (T_t u, T_t u)] \le \mathcal{E}(u,u).$$

(3) $G_\alpha(H) \subset D(\mathcal{E})$.

(4)

$$\mathcal{E}_\alpha(G_\alpha u, v) = (u, v), \qquad u \in H, v \in D(\mathcal{E}).$$

(5)

$$\frac{1}{t}(G_1 u - e^{-t} G_1 T_t u) = \frac{1}{t}(G_1 u - e^{-t} T_t G_1 u) \to u$$

as $t \to \infty$.

Proof. For (1) and (2),

$$\begin{aligned}\mathcal{E}(T_t u, T_t u) &= \int_0^\infty \lambda e^{-2\lambda t}(E(d\lambda)u,u)\\ &\le \int_0^\infty \frac{1}{2t}(1-e^{-2\lambda t})(E(d\lambda)u,u)\\ &\le \int_0^\infty \lambda(E(d\lambda)u,u)\\ &= \mathcal{E}(u,u).\end{aligned}$$

(3) and (4) have already been done. (5) is left as an exercise. □

Let

$$\mathcal{E}^{(t)}(u,v) = \frac{1}{t}(u - T_t u, v)$$

and

$$\mathcal{E}^{(\beta)}(u,v) = \beta(u - \beta G_\beta u, v).$$

Lemma 3.4 *(1)* $\mathcal{E}^{(t)}(u,u) \uparrow$ *as* $t \to 0$ *and*

$$D(\mathcal{E}) = \{u : \lim_{t\to 0} \mathcal{E}^{(t)}(u,u) < \infty\}, \mathcal{E}(u,v) = \lim_{t\to 0} \mathcal{E}^{(t)}(u,v).$$

(2) $\mathcal{E}^{(\beta)}(u,u) \uparrow$ *as* $\beta \to \infty$ *and*

$$D(\mathcal{E}) = \{u : \lim_{\beta\to\infty} \mathcal{E}^{(\beta)}(u,u) < \infty\}, \mathcal{E}(u,v) = \lim_{\beta\to\infty} \mathcal{E}^{(\beta)}(u,v).$$

Proof. (1)

$$\mathcal{E}^{(t)}(u,v) = \int_0^\infty \frac{1}{t}(1-e^{-\lambda t})(E(d\lambda)u,v) \to \int_0^\infty \lambda\,(E(d\lambda)u,v).$$

When $v = u$, use $\frac{1}{t}(1-e^{-\lambda t}) \uparrow \lambda$.

(2) Exercise. □

4 Markovian semigroups

A bounded linear operator S on $L^2(X, m)$ is Markovian if $0 \le u \le 1$, a.e., implies $0 \le Su \le 1$ a.e.

We suppose $\mathcal{E}$ is a closed symmetric form, X is a locally compact separable metric space, and m is a Radon measure.

Lemma 4.1 *Suppose S is a positive symmetric linear operator on L^2. There exists a unique symmetric measure σ on $X \times X$ such that*

$$(u, Sv) = \int u(x)v(y)\,\sigma(dx, dy).$$

If S is Markovian, $\sigma(X \times E) \le m(E)$.

Proof. Given σ, let $u = 1$ and $v = 1_E$,

$$\begin{aligned}\sigma(X \times E) &= (S1, 1_E) \le (1, 1_E)\\ &= \int 1 \cdot 1_E\, m(dx) = m(E).\end{aligned}$$

If $u_i, v_i \in C_0$ and $f(x, y) = \sum_{i=1}^{\ell} u_i(x)v_i(y)$, set

$$I(f) = \sum (u_i, Sv_j).$$

We claim that if $f \ge 0$, then $I(f) \ge 0$. Let $K = \cup_i \mathrm{supp}\, u_i$. Each u_i is uniformly continuous and K is compact. Write $K = \cup_{k=1}^p E_k$ and choose $z_k \in E_k$ such that

$$\sup_{x \in K} |u_i(x) - \widetilde{u}_i(x)| < \varepsilon,$$

where

$$\widetilde{u}_i(x) = \sum_{k=1}^{p} u_i(z_k) 1_{E_k}(x).$$

This is the analog of a step function approximation. So

$$|I(f) - \sum(\widetilde{u}_i, Sv_i)| < \varepsilon \sum (1_K, |Sv_i|)$$

and

$$\begin{aligned}\sum_{i=1}^{\ell}(\widetilde{u}_i, Sv_i) &= \sum_{i=1}^{\ell}\sum_{k=1}^{p}(u_i(z_k)1_{E_k}, Sv_i)\\ &= \sum_{k=1}^{p}\sum_{i=1}^{\ell} u_(z_k)(S1_{E_k}, v_i)\\ &= \sum_{k=1}^{p}(S1_{E_k}, f_k)\\ &= \sum_{k=1}^{p}(1_{E_k}, Sf_k),\end{aligned}$$

where $f_k(y) = f(z_k, y)$. $f \geq 0$, so $Sf_k \geq 0$, so $\sum_{i=1}^{\ell}(\widetilde{u}_i, Sv_i) \geq 0$.

Therefore I is a positive linear functional on the set of f of this form. Extend this to $C_0(X \times X)$ and use Riesz representation. □

Theorem 4.2 *Let $\mathcal{E}$ be a closed symmetric form on $L^2(X, m)$. The following are equivalent.*

(1) T_t is Markovian.

(2) αG_α is Markovian.

(3) $\mathcal{E}$ is Markovian.

(4) Unit contractions operate on $\mathcal{E}$.

(5) Normal contractions operate on $\mathcal{E}$.

Proof. : (1) implies (2): $G_\alpha u = \int_0^\infty e^{-\alpha t} T_t u\, dt$.

(2) implies (1): $T_t u = \lim e^{-t\beta} \sum (t\beta)^n (\beta G_\beta)^n u/n!$

(5) implies (4) and (4) implies (3): obvious.

So what is left is that (3) implies (2) and (2) implies (5).

(3) implies (2): Fix $\alpha > 0$ and $0 \leq u \leq 1$. Define

$$\psi(v) = \mathcal{E}(v, v) + \alpha\Big(v - \frac{u}{\alpha}, v - \frac{u}{\alpha}\Big), \qquad v \in D(\mathcal{E}).$$

Since $\mathcal{E}_\alpha(G_\alpha u, v) = (u, v)$,

$$\psi(G_\alpha u) + \mathcal{E}_\alpha(G_\alpha u - v, G_\alpha u - v) = \psi(v).$$

So G_α is the unique minimizer for ψ.

Suppose $\mathcal{E}$ is Markovian. Let

$$\widetilde{\phi}_\varepsilon(t) = \frac{1}{\alpha}\phi_{\alpha\varepsilon}(\alpha t), \qquad w = \widetilde{\phi}_\varepsilon(G_\alpha u).$$

Then $\mathcal{E}(w, w) \le \mathcal{E}(G_\alpha u, G_\alpha u)$.

$$|\widetilde{\phi}_\varepsilon(t) - s| \le |t - s|, \qquad s \in [0, 1/\alpha].$$

So

$$\Big| w(x) - \frac{u(x)}{\alpha} \Big| \le \Big| G_\alpha u(x) - \frac{u(x)}{\alpha} \Big|.$$

Hence

$$(w - \frac{u}{a}, w - \frac{u}{a}) \le (G_\alpha u - \frac{u}{a}, G_\alpha u - \frac{u}{a}).$$

So $\psi(w) \le \psi(G_\alpha u)$, hence $w = G_\alpha u$. Therefore $-\varepsilon \le G_\alpha u \le \frac{1}{\alpha} + \varepsilon$. Now use the fact that ε is arbitrary.

(2) implies (5): Assume αG_α is Markovian. There exists σ_α on $X \times X$ such that

$$(u, \alpha G_\alpha v) = \int u(x) v(y) \sigma_\alpha(dx, dy).$$

Then

$$\mathcal{E}^{(\beta)}(u, u) = \tfrac{1}{2}\beta \int (u(x) - u(y))^2 \, \sigma_\beta(dx, dy) \qquad (4.1)$$
$$+ \beta \int u(x)^2 \Big(1 - \frac{\sigma_\alpha(X \times dy)}{m(dy)} \Big) \, m(dx).$$

If the expression inside the big parentheses is $s_\alpha(x)$, then $0 \le s_\alpha(x) \le 1$. Now normal contractions operate on $\mathcal{E}$. □

A Dirichlet form is a Markovian closed symmetric form.

Proposition 4.3 *Let $\mathcal{E}$ be a Dirichlet form.*

(1) If $u, v \in D(\mathcal{E})$, then $u \vee v, u \wedge v, u \wedge 1 \in D(\mathcal{E})$.

(2) If $u, v \in D(\mathcal{E})$ are bounded, then $uv \in D(\mathcal{E})$ and

$$\mathcal{E}(uv, uv) \leq 2\|u\|_\infty^2 \mathcal{E}(v, v) + 2\|v\|_\infty^2 \mathcal{E}(u, u).$$

(3) If $u \in D(\mathcal{E})$, then $u_n = (-n \vee u) \wedge n$ is in $D(\mathcal{E})$ and $u_n \to u$ in $\mathcal{E}_1$ metric.

(4) If $u \in D(\mathcal{E})$, then $u^\varepsilon = u - [(-\varepsilon \vee u) \wedge u] \in D(\mathcal{E})$ and $u^\varepsilon \to u$ in $\mathcal{E}_1$ metric.

(5) $u_n \to u$ in $\mathcal{E}_1$ metric, $\phi(0) = 0$, $|\phi(t) - \phi(s)| \leq |t - s|$, then $\phi(u_n)$ converges weakly to $\phi(u)$ with respect to $\mathcal{E}_1$. If $\phi(u) = u$, then the convergence is strong.

Proof. (1) If $u \in D(\mathcal{E})$, then $|u| \in D(\mathcal{E})$ and $u \wedge 1 \in D(\mathcal{E})$. Now use

$$u \vee v = \tfrac{1}{2}(u + v + |u - v|), \qquad u \wedge v = \tfrac{1}{2}(u + v - |u - v|).$$

(2) If $|w(x) - w(y)| \leq |u_1(x) - u_1(y)| + |u_2(x) - u_2(y)|$ and $|w(x)| \leq |u_1| + |u_2|$, then $w \in D(\mathcal{E})$ and using (4.1),

$$\sqrt{\mathcal{E}(w, w)} \leq \sqrt{\mathcal{E}(u_1, u_1)} + \sqrt{\mathcal{E}(u_2, u_2)}.$$

Set $w = uv$ with $u_1 = \|u\|_\infty v$, $u_2 = \|v\|_\infty u$.

(3) u_n is a normal contraction of u, so $\mathcal{E}_1(u_n, u_n) \leq \mathcal{E}_1(u, u)$.

$$\mathcal{E}_1(u_n, G_1 v) = (u_n, v) \to (u, v) = \mathcal{E}_1(u, G_1 v).$$

$G_1(L^2)$ is dense in $D(\mathcal{E})$, so $u_n \xrightarrow{w} u$ with respect to the $\mathcal{E}_1$ norm. Then

$$\mathcal{E}_1(u_n - u, u_n - u) \leq 2\mathcal{E}_1(u, u) - 2\mathcal{E}_1(u_n, u) \to 0.$$

The proofs of (4) and (5) are the same as that of (3). □

A family P_t of Markovian kernels is a Markovian transition function if $P_t P_s u = P_{t+s} u$. A resolvent R_α is a Markovian resolvent if R_α is Markovian and the resolvent equation holds:

$$R_\alpha u - R_\beta u + (\alpha - \beta) R_\alpha R_\beta = 0.$$

A kernel K is m-symmetric if

$$\int u(Kv)\,dm = \int (Ku)v\,dm.$$

If K is m-symmetric and Markovian,

$$Ku(x)^2 \le K1(x)Ku^2(x)$$

by Cauchy-Schwarz, so

$$\int (Ku)^2\,dm \le \int u^2\,dm.$$

A semigroup T_t is conservative if $T_t 1 = 1$, a.e.

Exercise: if $1 \in D(\mathcal{E})$ and $\mathcal{E}(1,1) = 0$, then T_t is conservative.

5 Mosco convergence

If $u \notin D(\mathcal{E})$, set $\mathcal{E}(u,u) = \infty$.

$\mathcal{E}_n$ converges to $\mathcal{E}$ in the sense of Mosco if

(1) whenever $u_n \xrightarrow{w} u$, then $\liminf \mathcal{E}_n(u_n, u_n) \ge \mathcal{E}(u,u)$.

(2) If $u \in H$, there exists $u_n \xrightarrow{s} u$ such that $\limsup \mathcal{E}_n(u_n, u_n) \le \mathcal{E}(u,u)$.

Theorem 5.1 *If $\mathcal{E}_n$ converges to $\mathcal{E}$ in the sense of Mosco, then for all β, $G^n_\beta \xrightarrow{s} G_b$.*

Proof. We first note that $u = G_\lambda f$ minimizes

$$\mathcal{E}(u,u) + \lambda(u,u) - 2(f,u).$$

To see this, look at $u + \varepsilon v$. Expanding,

$$\begin{aligned}\mathcal{E}_\lambda(u,u) - 2(f,u) &\le \mathcal{E}_\lambda(u+\varepsilon v, u+\varepsilon v) - 2(f, u+\varepsilon v)\\ &= \mathcal{E}_\lambda(u,u) = 2\varepsilon\mathcal{E}_\lambda(u,v) - 2(f,u)\\ &\quad - 2\varepsilon(f,v) + O(\varepsilon^2).\end{aligned}$$

So

$$2\mathcal{E}_\lambda(u,v) - 2(f,u) = 0$$

for all $v \in D(\mathcal{E})$. Therefore $u = G_\lambda f$.

Let $u_n = G^n_\beta f$. $\|u_n\| \le \frac{1}{\lambda}\|f\|$, so there exists a subsequence, denoted again by u_n, such that u_n converges weakly, say to u. We claim $u = G_\beta f$.

By (1),

$$\liminf \mathcal{E}_n(u_n,u_n) \ge \mathcal{E}(u,u).$$

If $v \in H$, take $v_n \to v$ such that $\mathcal{E}_n(v_n,v_n) \to \mathcal{E}(v,v)$. Then because u_n is the minimizer,

$$\mathcal{E}_n(u_n,u_n) + \beta(u_n,u_n) - 2(f,u_n) \le \mathcal{E}_n(v_n,v_n) + \beta(v_n,v_n) - 2(f,v_n).$$

Let $n \to \infty$. Then $\liminf(u_n,u_n) \ge (u,u)$. So

$$\mathcal{E}(u,u) + \beta(u,u) - 2(f,u) \le \mathcal{E}(v,v) + \beta(v,v) - 2(f,v).$$

So u is the minimizer, and hence $u = G_\beta f$.

$$(u_n,u_n) = \frac{1}{\beta}((f,u_n) - \mathcal{E}_n(u_n,u_n)),$$

so

$$\limsup(u_n,u_n) \le \frac{1}{\beta}((f,u) - \mathcal{E}(u,u)) = (u,u).$$

Since $u_n \xrightarrow{w} u$, then $(u_n,u) \to (u,u)$. Therefore

$$\begin{aligned}\limsup(u_n - u, u_n - u) &= \limsup(u_n,u_n) - 2\lim(u,u_n) + (u,u)\\ &\le (u,u) - 2(u,u) + (u,u) = 0.\end{aligned}$$

□

The theorem turns out to be if and only if.

6 Capacities

We now assume $\mathcal{E}$ is regular and we write $\mathcal{F}$ for $D(\mathcal{E})$.

If A is open, define

$$\mathrm{Cap}(A) = \inf\{\mathcal{E}_1(u,u) : u \geq 1 \text{ a.e. on } A\}.$$

If $A \subset X$, let $\mathrm{Cap}\, A = \inf\{\mathrm{Cap}\, B :_B \text{ open}, A \subset B\}$.

Lemma 6.1 *(1) If A is open and $\mathrm{Cap}\, A \neq \infty$, then there exists a unique element e_A of $\mathcal{F}$ such that $\mathcal{E}_1(e_A, e_A) = \mathrm{Cap}\, A$.*

(2) $0 \leq e_A \leq 1$, a.s., and $e_A = 1$ a.e. on A.

(3) e_A is the unique element of $\mathcal{F}$ satisfying $e_A = 1$ a.e. on A and $\mathcal{E}_1(e_A, v) \geq 0$ for all $v \in \mathcal{F}$ such that $v \geq 0$ a.e. on A.

(4) If $v \in \mathcal{F}$, $v = 1$ a.e. on A, then $\mathcal{E}_1(e_A, v) = \mathrm{Cap}\, A$.

(5) If A and B are open and $\mathrm{Cap}\, A, \mathrm{Cap}\, B < \infty$, then $e_A \leq e_B$, a.e.

Proof. (1) Let $\mathcal{L}_A = \{u \in \mathcal{F} : u \geq 1 \text{ a.e. on } A\}$. $\mathcal{L}_A$ is convex, closed, and a subset of $\mathcal{F}$. Therefore there exists a unique minimizing element minimizing the distance to 0.

(2) $u = (0 \vee e_A) \wedge 1 \in \mathcal{L}_A$, and $\mathcal{E}_1(u,u) \leq \mathcal{E}_1(e_A, e_A) = \mathrm{Cap}\, A$. So $u = e_A$.

(3) $e_A + \varepsilon v \in \mathcal{L}_A$, and $\mathcal{E}_1(e_A + \varepsilon v, e_A + \varepsilon v) \geq \mathcal{E}_1(e_A, e_A)$. So $\mathcal{E}_1(e_A, v) = 0$.

Conversely, if u is another such function, then $u \in \mathcal{L}_A$. So

$$\mathcal{E}_1(w,w) = \mathcal{E}(u + (w-u), u + (w-u)) \geq \mathcal{E}_1(u,u)$$

for all w. Therefore $u = e_A$.

(4) Note $e_A - v = 0$ on A. So $\mathcal{E}_1(e_A, e_A - v) \geq 0$ and $\mathcal{E}_1(e_A, v - e_A) \geq 0$, and then

$$\mathcal{E}_1(e_A, v) = \mathcal{E}_1(e_A, e_A) = \mathrm{Cap}\, A.$$

(5) Starting from $\mathcal{E}_1(|u|, |u|) \leq \mathcal{E}_1(u,u)$ and writing $u = u^+ - u^-$ and $|u| = u^+ + u^-$, we obtain $\mathcal{E}_1(u^+, u^-) \leq 0$ for all $u \in \mathcal{F}$. $e_A - (e_A \wedge e_B) = (e_A - e_B)^+ = 0$ a.e on A. Then $\mathcal{E}_1(e_A, (e_A - e_B)^+) = 0$ and

$$\begin{aligned}\mathcal{E}_1(e_A - e_A \wedge e_B, e_A - e_A \wedge e_B) &= \mathcal{E}_1(-e_A \wedge e_B, (e_A - e_B)^+) \\ &= \mathcal{E}_1((e_A - e_B)^-, (e_A - e_B)^+) - \mathcal{E}_1(e_B, (e_A - e_B)^+) \leq 0.\end{aligned}$$

Therefore $e_A = e_A \wedge e_B$ a.e. □

Lemma 6.2 *(1) If A and B are open and $A \subset B$, then* $\operatorname{Cap} A \leq \operatorname{Cap} B$.

(2) $\operatorname{Cap}(A \cup B) + \operatorname{Cap}(A \cap B) \leq \operatorname{Cap} A + \operatorname{Cap} B$, *if A and B are open.*

(3) If $A_n \uparrow$ and open, then $\operatorname{Cap}(\cup A_n) = \sup \operatorname{Cap}(A_n)$.

Proof. (1) is clear.

(2)

$$\begin{aligned}
\operatorname{Cap}(A \cup B) + \operatorname{Cap}(A \cap B) &\leq \mathcal{E}_1(e_A \vee e_B, e_A \vee e_B) + \mathcal{E}_1(e_A \wedge e_B, e_A \wedge e_B) \\
&= \tfrac{1}{2}\mathcal{E}_1(e_A + e_B, e_A + e_B) + \tfrac{1}{2}\mathcal{E}_1(|e_A - e_B|, |e_A - e_B|) \\
&\leq \mathcal{E}_1(e_A, e_A) + \mathcal{E}_1(e_B, e_B) \\
&= \operatorname{Cap} A + \operatorname{Cap} B.
\end{aligned}$$

We used here $\mathcal{E}_1(|e_a - e_B|, |e_A - e_B|) \leq \mathcal{E}_1(e_A - e_B, e_A - e_B)$.

(3) Suppose $\sup_n \operatorname{Cap} A_n < \infty$. If $n > m$,

$$\mathcal{E}_1(e_{A_n} - e_{A_m}, e_{A_n} - e_{A_m}) = \operatorname{Cap} A_n - \operatorname{Cap} A_m,$$

where we used (4) from the preceding lemma. So e_{A_n} converges to u with respect to $\mathcal{E}_1$. $u = 1$ a.e. on $A = \cup A_n$. If $v \geq 0$ a.e. on A, $\mathcal{E}_1(u, v) = \lim \mathcal{E}_1(e_{A_n}, v) \geq 0$. So $u = e_A$. Then

$$\sup_n \operatorname{Cap} A_n = \lim \mathcal{E}_1(e_{A_n}, e_{A_n}) = \mathcal{E}_1(u, u) = \operatorname{Cap} A.$$

The case where $\sup_n \operatorname{Cap} A_n = \infty$ is left as an exercise. □

The following theorem is a consequence of Choquet's capacibility theorem.

Theorem 6.3 *(1) If $A \subset B$, then* $\operatorname{Cap} A \leq \operatorname{Cap} B$.

(2) If $A_n \uparrow$, then $\operatorname{Cap}(\cup A_n) = \sup \operatorname{Cap} A_n$.

(3) If A_n are compact and $A_n \downarrow$, then $\operatorname{Cap}(\cap A_n) = \inf_n \operatorname{Cap} A_n$.

(4) $\operatorname{Cap} A = \sup\{\operatorname{Cap} K : K \subset A, K \textit{ compact}\}$.

By the definition, $m(A) \leq \operatorname{Cap} A$ if A is open. So if $\operatorname{Cap} A = 0$, then $A = 0$ a.e. A property holds q.e. (quasi everywhere) if there exists N such that $\operatorname{Cap} N = 0$ and the property holds for all $x \in X \setminus N$.

Some facts:

(1) If $X = \mathbb{R}^2$ and A is the real axis, then $m(A) = 0$ but $\operatorname{Cap} A \neq 0$.

(2) A capacity is not necessarily a measure.

A function u is quasi-continuous if given $\varepsilon > 0$ there exists $G \subset X$ open such that $\operatorname{Cap} G < \varepsilon$ and $u\,|_{X \setminus G}$ is continuous.

If F_K is closed, $F_K \uparrow$, and $\operatorname{Cap}(X \setminus F_k) \to 0$, then $\{F_k\}$ is called a nest.

Proposition 6.4 *If u is quasi-continuous and $u \geq 0$ a.e., then $u \geq 0$ q.e.*

Proof. Choose $G_n \downarrow$ such that $\operatorname{Cap} G_n < 2^{-n}$ and u restricted to G^c is continuous. Let $F_n = G_n^c$ and F'_n the support of the measure $1_{F_n}\,dm$. F'_n is the smallest closed set whose complement is $1_{F_n}\,dm$-negligible. So $F'_n \subset F_n$ and

$$m((F'_n)^c - F_n^c) = \int_{(F'_n)^c} 1_{F_n}\,dm = 0.$$

We claim $\operatorname{Cap}(F'_n)^c = \operatorname{Cap} F_n^c$. To see this, both F_n^c and $(F'_n)^c$ are open, and if u is such that $u \in \mathcal{L}_{F_n^c}$, then $u \in \mathcal{L}_{(F'_n)^c}$ and vice versa. So from the definition of capacity, the two sets have the same capacity.

Suppose $u(x) < 0$ for some $x \in F'_k$. There exists a neighborhood $U(x)$ such that $u(y) < 0$ for $y \in U(x) \cap F'_k$. But then $m(U(x) \cap F'_k) > 0$, a contradiction to $u \geq 0$ a.e. □

v is a quasi-continuous modification of u if v is quasi-continuous and $v = u$ a.e. We sometimes write $v = \widetilde{u}$.

Theorem 6.5 *Suppose $\mathcal{E}$ is regular. If $u \in \mathcal{F}$, then u has a quasi-continuous modification.*

Proof. We show

$$\operatorname{Cap}\{x : |u(x)| > \lambda\} \leq \frac{1}{\lambda^2}\mathcal{E}_1(u,u), \qquad u \in \mathcal{F} \cap C(X). \tag{1}$$

To see this, let G be this set. Since $u \in C(X)$, G is open. $|u|/\lambda \in \mathcal{L}_G$. So

$$\operatorname{Cap} G \leq \frac{1}{\lambda^2}\mathcal{E}_1(|u|,|u|) \leq \frac{1}{\lambda^2}\mathcal{E}_1(u,u).$$

Now, given $u \in \mathcal{F}$, choose $u \in \mathcal{F} \cap C_0$ such that $u_n \to u$ in $\mathcal{E}_1$ metric. By taking a subsequence, we may assume

$$\mathcal{E}_1(u_{k+1} - u_k, u_{k+1} - u_k) \le 2^{-3k}.$$

By (1), $\operatorname{Cap} G_k \le 2^{-k}$, where $G_k = \{x : |u_{k+1} - u_k| > 2^{-k}\}$. Let $F_k = \cap_{\ell=k}^{\infty} G_\ell^c$. $\{F_k\}$ is a nest.

If $x \in F_k$ and $n, m \ge N \ge k$,

$$|u_n(x) - u_m(x)| \le \sum_{j=N+1}^{\infty} |u_{j+1}(x) - u_j(x)| \le 2^{-N}.$$

So $u_n |_{F_k}$ converges uniformly. Let $\widetilde{u}$ be the limit. Then $\widetilde{u} |_{F_k}$ is continuous and $u = \widetilde{u}$, a.e. □

Corollary 6.6 *The inequality (1) holds for all $u \in D(\mathcal{E})$.*

Proof. Take limits. □

7 Measures of finite energy integrals

A positive Radon measure μ in X has finite energy if

$$\int |v(x)| \, \mu(dx) \le c\sqrt{\mathcal{E}_1(v, v)}, \qquad v \in \mathcal{F} \cap C_0,$$

for some c.

μ is of finite energy integral if and only if: for all $\alpha > 0$, there exists a unique function $U_\alpha \mu \in \mathcal{F}$ such that

$$\mathcal{E}_\alpha(U_\alpha \mu, v) = \int v(x) \, \mu(dx)$$

for all $v \in \mathcal{F} \cap C_0$.

$U_\alpha \mu$ is called a potential.

Assume $\mathcal{E}$ is a regular Dirichlet form.

u is α-excessive with respect to T_t if $u \ge 0$ and $e^{-\alpha t} T_t u \le u$ a.e. for all t.

Theorem 7.1 *The following are equivalent.*

(1) u is an α-potential.

(2) u is α-excessive

(3) $u \geq 0$ and $\beta G_{\alpha+\beta} u \leq u$ a.e. for all β.

(4) $\mathcal{E}_\alpha(u, v) \geq 0$ for all $v \in \mathcal{F}$ with $v \geq 0$, a.e.

(5) $\mathcal{E}_\alpha(u, v) \geq 0$ for all $v \in \mathcal{F} \cap C_0$ with $v \geq 0$.

Proof. (2) implies (3): multiply $e^{-\alpha t} T_t u \leq u$ by $\beta e^{-\beta t}$ and integrate over t.

(3) implies (4): $\mathcal{E}_\alpha(u, v) = \lim_\beta \beta(u - \beta G_{\beta+\alpha} u, v)$.

(4) implies (2): u is the unique element in

$$\mathcal{L}_u = \{w \in \mathcal{F} : w \geq u \text{ a.e.}\}$$

minimizing $\mathcal{E}_\alpha(w, w)$. To see this, use

$$\begin{aligned} \mathcal{E}_\alpha(w, w) = \mathcal{E}_\alpha(w - u, w - u) + 2\mathcal{E}_\alpha(w - u, u) \\ + \mathcal{E}_\alpha(u, u) \\ \geq \mathcal{E}_\alpha(u, u). \end{aligned}$$

$|u| \in \mathcal{L}_u$ so $u = |u| \geq 0$. Moreover

$$G_\alpha v - e^{-\alpha t} T_t G_\alpha v = \int_0^t e^{-\alpha s} T_s v \, ds \geq 0,$$

so

$$(u - e^{-\alpha t} T_t u, v) = (u, v - e^{-\alpha t} T_t v) = \mathcal{E}_\alpha(u, G_\alpha v - e^{-\alpha t} T_t G_\alpha v) \geq 0.$$

So u is α-excessive.

That (1) implies (5) and that (4) implies (5) are trivial.

(5) implies (4): If $v \geq 0$ and $v \in \mathcal{F}$, take $v_n \in \mathcal{F} \cap C_0$ with $v_n \to v$ in $\mathcal{E}_1$ metric. Then $v_n^+ \to v^+$ in $\mathcal{E}_1$ metric, so

$$\mathcal{E}_\alpha(u, v) = \lim \mathcal{E}_\alpha(u, v_n^+) \geq 0.$$

(5) implies (1): For simplicity, let us suppose X is compact. Let $I(v) = \mathcal{E}_\alpha(u,v)$ for $v \in \mathcal{F} \cap C_0$. The function 1 is in $\mathcal{F}$, so $\mathcal{E}_\alpha(1,1) < \infty$. Then

$$|I(v)| \leq c\|v\|_\infty$$

if $v \in \mathcal{F} \cap C_0$. I can be extended to a positive linear functional on $C_0(X)$. u is the α-potential of a positive Radon measure. □

Corollary 7.2 *If $u_1, u_2 \in \mathcal{F}$ are α-potentials, so are $u_1 \wedge u_2$ and $u_1 \wedge 1$.*

Proof. True for α-excessive functions. □

Let S_0 be collection of positive Radon measures of finite energy integral.

Lemma 7.3 *If $\mu \in S_0$ and $\alpha > 0$, let*

$$g_n = n(U_\alpha\mu - n g_{n+\alpha}(U_\alpha\mu)).$$

Then $g_n\, dm \xrightarrow{w} \mu$ and $G_\alpha g_n \to U_\alpha\mu$ weakly with respect to the $\mathcal{E}_\alpha$ metric.

Proof. Let $u = U_\alpha\mu$. So $g_n \geq 0$, a.e.

$$\mathcal{E}_\alpha(G_\alpha g_n, v) = (g_n, v) = n(u - nG_{n+\alpha}u, v) \to \mathcal{E}_\alpha(u,v)$$

if $v \in \mathcal{F}$. In particular, $(g_n, v) \to \int v\, d\mu$ for all $v \in \mathcal{F} \cap C_0$. □

Lemma 7.4 *If $\mu \in S_0$ and $\operatorname{Cap} A = 0$, then $\mu(A) = 0$.*

Proof. If G is open, since $e_G \geq 1$ on G,

$$\begin{aligned}
\mu(G) &= \int 1_G\, d\mu \\
&\leq \liminf \int g_n(x)\, m(dx) \leq \lim(g_n, e_G) \\
&= \lim \mathcal{E}_1(G_1 g_n, e_G) \\
&= \mathcal{E}_1(U_1\mu, e_G) \leq \sqrt{\mathcal{E}_1(U_1\mu, U_1\mu)}\sqrt{\mathcal{E}_1(e_G, e_G)} \\
&= \sqrt{\mathcal{E}_1(U_1\mu, U_1\mu)}(\operatorname{Cap} G)^{1/2}.
\end{aligned}$$

□