# Math 3410, Fall 2012
# Second semester differential equations

## 1 Some review

### 1.1 First order equations

We will need to know how to do separable equations and linear equations. We'll also review exact and homogeneous equations.

A separable equation is one like

$$\frac{dy}{dx} = xy^2.$$

We can rewrite this as

$$\frac{dy}{y^2} = x\,dx.$$

Integrating both sides,

$$-\frac{1}{y} = \tfrac{1}{2}x^2 + c,$$

so

$$y = -\frac{1}{\frac{1}{2}x^2 + c}.$$

To solve for $c$ we need extra information. If, for example, $y(0) = 1$, we substitute and get

$$1 = -\frac{1}{c},$$

or $c = -1$. Our complete solution is then

$$y = -\frac{1}{\frac{1}{2}x^2 - 1}.$$

For another example, if

$$\frac{dy}{dx} = \frac{x}{1 + y^2},$$

then

$$(1+y^2)\,dy = x\,dx,$$

and integrating gives

$$y + y^3 = x^2 + c.$$

(This equation is difficult to solve for $y$ in terms of $x$.)

In general, a separable equation has the form

$$\frac{dy}{dx} = f(x)g(y).$$

A linear equation is one like

$$y' + \frac{2}{x}y = x.$$

One multiplies by an integrating factor

$$p = e^{\int \frac{2}{x}} = e^{2\ln x} = x^2$$

to get

$$x^2y' + 2xy = x^3,$$

or

$$(x^2y)' = x^3,$$

which leads to

$$x^2y = \tfrac{1}{4}x^4 + c,$$

and then

$$y = \tfrac{1}{4}x^2 + \frac{c}{x^2}.$$

Another example:

$$y' + 2xy = x.$$

Multiply both sides by $e^{x^2}$. Then

$$e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}.$$

The left hand side is the same as

$$(e^{x^2}y)'.$$

So integrating,

$$e^{x^2}y = \int xe^{x^2}\,dx = \tfrac{1}{2}e^{x^2} + c,$$

and then

$$y = \tfrac{1}{2} + ce^{-x^2}.$$

The general linear equation is of the form

$$y' + f(x)y = g(x).$$

We multiply both sides by

$$p(x) = e^{\int f(x)\,dx}.$$

Then $p'(x) = f(x)p(x)$ and we get

$$(p(x)y)' = p(x)y' + p'(x)y = p(x)g(x).$$

When the linear equation has constant coefficients and is homogeneous (i.e., the right hand side is 0), things are much easier. To solve

$$y' - 4y = 0,$$

we guess a solutions of the form $y = e^{rx}$, so $y' = re^{rx}$. Then

$$re^{rx} - 4e^{rx} = 0,$$

or $r = 4$, and therefore the solution is

$$y = ce^{4x}.$$

To identify $c$, one needs an initial condition, e.g., $y(0) = 2$. Then

$$2 = ce^{4\cdot 0} = c,$$

so we then have

$$y = 2e^{4x}.$$

For non-homogeneous equations, such as

$$y' - 4y = e^{3x},$$

one way to solve it is to solve the homogeneous equation $y' - 4y = 0$, and then use

$$y = ce^{4x} + y_p,$$

where $y_p$ is a particular solution. One way to find a particular solution is to make an educated guess. If we guess $y_p = Ae^{3x}$, then we have

$$y_p' - 4y_p = 3Ae^{3x} - 4Ae^{3x},$$

and this will equal $e^{3x}$ if $A = -1$. We conclude the solution to the non-homogeneous equation is

$$y = ce^{4x} - e^{3x}.$$

An example of an exact equation is

$$(x + y)y' + (x^2 + y) = 0.$$

We rewrite this as

$$(x^2 + y)\,dx + (x + y)\,dy = 0. \tag{1.1}$$

This leads to

$$xy + \frac{y^2}{2} + \frac{x^3}{3} = c. \tag{1.2}$$

One way to get (1.2) is by inspection. A more detailed way is to try to write (1.1) as

$$df(x, y) = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

and then integrate. If we try to do this, we have $\partial f/\partial x = x^2 + y$, or $f = x^3/3 + xy + h(y)$ for any function $h$. Then $\partial f/\partial y = x + h'(y) = x + y$, which implies $h(y) = y^2/2$. Thus $f(x, y)$ equals the left hand side of (1.2).

In general, an exact equation is one of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

with

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Under this condition there is a function $f(x, y)$ such that

$$M(x, y) = \frac{\partial f}{\partial x}(x, y) \qquad \text{and} \qquad N(x, y) = \frac{\partial f}{\partial y}(x, y).$$

So we can use the chain rule to rewrite the equation as

$$df(x,y) = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

and get the solution by integrating both sides.

Finally we look at homogeneous equations. An example is

$$y' = \frac{y}{x} + \frac{y^2}{x^2}.$$

In general a homogeneous equation is one where

$$y' = f(y/x).$$

To solve, we let $v = y/x$, so $y = xv$, and then

$$y' = xv' + v.$$

If we use this in our example, we get

$$xv' + v = v + v^2,$$

or $xv' = v^2$, which is separable. (It is always the case that one gets a separable equation.) So

$$\frac{1}{v^2}\,dv = \frac{1}{x}\,dx,$$

or

$$-\frac{1}{v} = \ln x + c,$$

which can be solved as

$$v = -\frac{1}{\ln x + c}.$$

Now recall that $v = y/x$, or

$$y = -\frac{x}{\ln x + c}.$$

### 1.2 Series

From calculus we have the Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

If $i = \sqrt{-1}$, then $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and then the powers of $i$ cycle again and again. Substituting and doing some algebra shows that

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} \\ &\quad + i\frac{x^5}{5!} - \frac{x^6}{6!} + \cdots \\ &= \Big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\Big) \\ &\quad + i\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\Big), \end{aligned}$$

or we end up with

$$e^{ix} = \cos x + i \sin x.$$

# 2 Second order linear equations

### 2.1 Applications

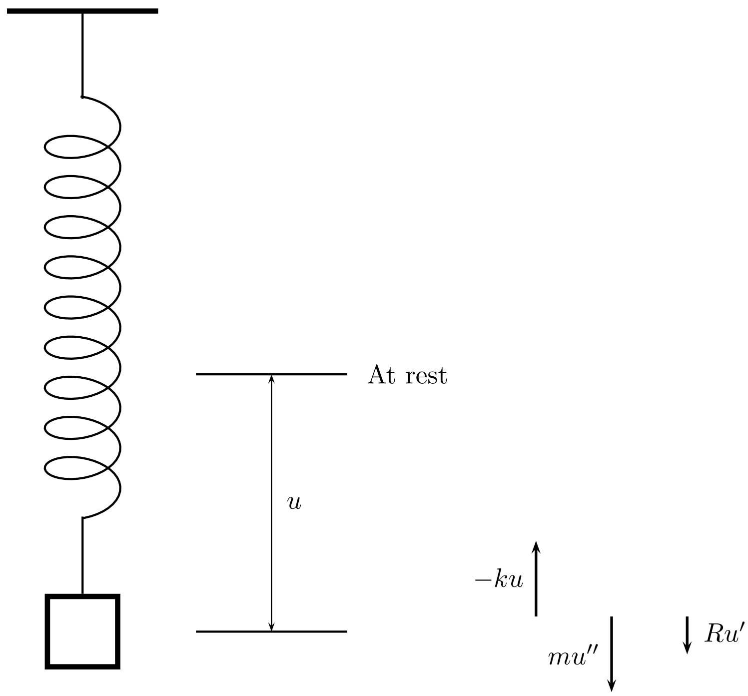
First consider a spring hung from the ceiling with a weight hanging from it. Let $u$ be the distance the weight is below equilibrium. There is a restoring force upwards of amount $-ku$ by Hooke's law. There is damping resistance against the motion, which is $-Ru'$. And the net force is related to acceleration by Newton's laws, so

$$ku - Ru' = F = mu''.$$

This leads to

$$mu'' + Ru' + ku = 0.$$

If there is an external force acting on the spring, then the right hand side is replaced by $F(t)$.

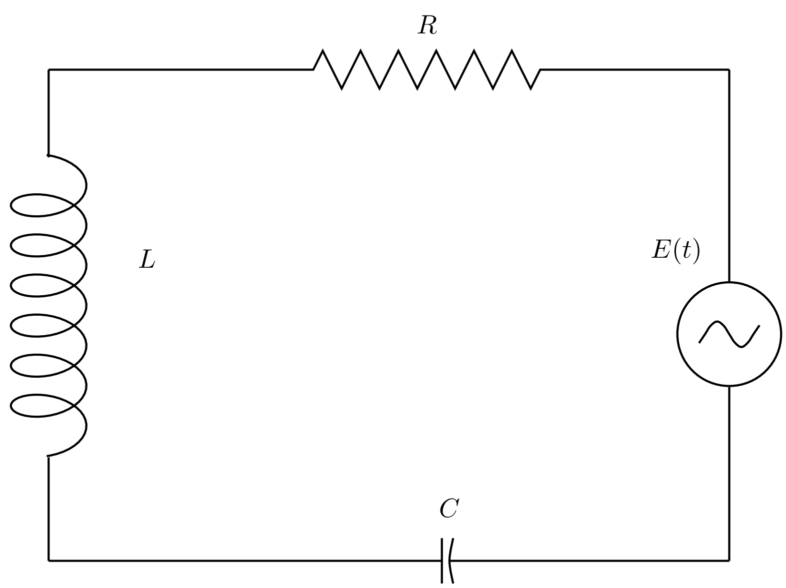

At rest
$u$
$-ku$
$mu''$
$Ru'$

The second example is that of a circuit with a resistor, inductance coil, and capacitor hooked up in series. Let $I$ be the current, $Q$ the charge, $R$ the resistance, $L$ the inductance, and $C$ the capacitance. We know that $I = dQ/dt$. The voltage drop across the resistor is $IR$, across the capacitor $Q/C$, and across the inductance coil $L\frac{dI}{dt}$. So if $E(t)$ is the potential put into the current,

$$E(t) = LQ'' + RQ' + \frac{1}{C}Q.$$

Sometimes this is differentiated to give

$$E'(t) = LI'' + RI' + \frac{1}{C}I.$$


$R$
$L$
$E(t)$
$C$

## 2.2 Second order linear equations with constant coefficients, homogeneous

Let's look at an example:

$$y'' - 5y' + 4y = 0.$$

From Math 2410, we know a way of solving this. Let $v = y'$, and this one equation becomes a system

$$\begin{aligned} y' &= v; \\ v' &= 5v - 4y. \end{aligned}$$

We then set up matrices, where

$$X = \begin{pmatrix} y \\ v \end{pmatrix},$$

$$A = \begin{pmatrix} -4 & 5 \\ 0 & 1 \end{pmatrix},$$

and the equation is

$$X' = AX.$$

We assume

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and that our solution is of the form

$$X = We^{rt}$$

for some $r$, $w_1$, and $w_2$. We will review this method later when we want to generalize it, but let's look at an easier method. (We let our independent variable be $t$ instead of $x$ because in our applications we often want functions to be a function of time.)

Let's guess our solution is of the form

$$y = e^{rt}$$

for some $r$. Then

$$y' = re^{rt}, \qquad y'' = r^2 e^{rt}.$$

So our equation becomes

$$e^{rt}(r^2 - 5r + 4) = 0.$$

This factors as $(r-4)(r-1) = 0$, or $r = 1, 4$. The general solution to our equation is then

$$y = c_1 e^{4t} + c_2 e^t.$$

Here is another example:

$$y'' + 8y' - y = 0.$$

Assuming $y = e^{rt}$ then leads to

$$r^2 + 8r - 1 = 0,$$

or

$$r = \frac{-8 \pm \sqrt{68}}{2} = -4 \pm \sqrt{17}.$$

Our general solution is then

$$y = c_1 e^{(-4+\sqrt{17})t} + c_2 e^{(-4-sqrt17)t}.$$

This method works except for two cases. If $r_1 = r_2$, we do not get two solutions. Thus, for example, the equation

$$y'' - 4y' + 4y = 0$$

leads to $r = 2$. In this case we have as a solution

$$y = c_1 e^{2t} + c_2 t e^{2t}.$$

We differentiate to check that this works.

Secondly we might have $r_1 = a + bi, r_2 = a - bi$. For example, if we have the equation

$$y'' - y' + 6y = 0,$$

we get $r^2 - r + 6 = 0$, whose solutions are

$$r = \frac{1 \pm \sqrt{-23}}{2} = \frac{1}{2} \pm i\frac{\sqrt{23}}{2}.$$

Actually in this case we proceed and everything works out eventually. It goes like this. Here $a = 1/2$, $b = \sqrt{23}/2$.

$$e^{(a+bi)t} = e^{at}e^{bti} = e^{at}(\cos bt + i \sin bt),$$

and similarly

$$e^{(a-bi)t} = e^{at}(\cos bt - i \sin bt)$$

since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin(\theta)$. Then

$$\begin{aligned} y &= c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t} \\ &= c_1 e^{at} \cos bt + c_1 i e^{at} \sin bt + c_2 e^{at} \cos bt - c_2 i e^{at} \sin bt \\ &= (c_1 + c_2) e^{at} \cos bt + (c_1 i - c_2 i) e^{at} \sin bt \\ &= K_1 e^{at} \cos bt + K_2 e^{at} \sin bt. \end{aligned}$$

Once we see how this goes, we immediately can go to the last line without going through the derivation. In our example, our solution is

$$y = K_1 e^{t/2} \cos(\sqrt{23}t/2) + K_2 e^{t/2} \sin(\sqrt{23}t/2).$$

## 2.3 Constants

To get the values of $c_1$ and $c_2$ we need extra information. In an initial value problem, we are given $y(t_0)$ and $y'(t_0)$ for some $t_0$. For example, consider

$$y'' + 4y = 0, \qquad y(0) = 3, y'(0) = 4.$$

We solve $r^2 + 4 = 0$, or $r = \pm 2i$. So

$$y = c_1 \cos 2t + c_2 \sin 2t.$$

Then

$$3 = y(0) = c_1 \cdot 1 + c_2 \cdot 0,$$

or $c_1 = 1$. Differentiating,

$$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t,$$

and substituting in $t = 0$, we get $c_2 = 2$.

One could also look at the same ODE but instead of initial values, suppose we are given boundary values: $y(0) = 0, y(1) = 3$. Then as before $c_1 = 0$, which leads to

$$y = c_2 \sin 2t.$$

Putting in $t = 1$, we get $c_2 = 3/\sin 2$.

There is no analog of boundary value problems for first order equations.

## 2.4 Method of undetermined coefficients

When solving

$$y'' - 5y' + 4y = e^{3t},$$

the general solution is $y_h + y_p$, where $y_h$ is the general solution to the homogeneous equation and $y_p$ is a particular solution to the non-homogeneous equation. One way to get a particular solution is the method of undetermined coefficients: try $y = Ae^{3t}$. Then

$$y' = 3Ae^{3t}, \qquad y'' = 9Ae^{3t},$$

and substituting gives

$$9Ae^{3t} - 15Ae^{3t} + 4e^{3t} = e^{3t}.$$

So $A = -\frac{1}{2}$, and $y_p = -\frac{1}{2}e^{3t}$, and thus

$$y = c_1 e^{4t} + c_2 e^t - \frac{1}{2}e^{3t}.$$

To find $c_1, c_2$, one waits until one has the most general solution before using initial or boundary values.

If the right hand side were $2e^{4t}$, this doesn't work, but one could try

$$y_p = Ate^{4t}$$

to get a particular solution. We get $y' = 4Ate^{4t} + Ae^{4t}$ and $y'' = 16Ate^{4t} + 8Ae^{4t}$. Substituting, we have

$$0 \cdot Ate^{4t} + 3Ae^{4t} = e^{4t},$$

so $A = 2/3$.

If on the right hand side there was $\cos t$, one needs to try $A\cos t + B\sin t$. One could also have $e^{4t} + \cos t + 2e^{2t}$ on the right hand side, for example, and one finds a particular solution for each piece, and then adds.

## 2.5 Euler equation

An equation like

$$x^2y'' + 4xy' - 4y = 0$$

is called an Euler equation. Try $y = x^r$ as a trial solution. Then

$$y' = rx^{r-1}, \qquad y'' = r(r-1)x^{r-2}.$$

Substituting,

$$r(r-1)x^1x^{r-2} + 4xrx^{r-1} - 4x^r = 0,$$

or

$$r(r-1) + 4r - 4 = 0,$$

or $r = -4, 1$. Then the general solution is

$$y = c_1x^{-4} + c_2x^1.$$

Again there are variations when $r_1 = r_2$ or when $r$ is complex. If $r_1 = r_2$, the solution is

$$c_1x^r + c_2x^r \ln x.$$

If $r = a \pm bi$, the solution is

$$y = c_1x^a \cos(b \ln x) + c_2x^a \sin(b \ln x).$$

For another example consider

$$x^2y'' + 4xy' - 6y = 0.$$

We use the same technique as in the previous example and have

$$r(r-1) + 4r - 6 = 0,$$

or $r^2 + 3r - 6 = 0$. The solutions are

$$\frac{-3 \pm \sqrt{33}}{2},$$

and thus the solution to the differential equation is

$$y = c_1x^{-\frac{3}{2}+\frac{\sqrt{33}}{2}} + c_2x^{-\frac{3}{2}-\frac{\sqrt{33}}{2}}.$$

# 3 Series solutions

We show how to use Taylor series to solve equations such as

$$y'' + xy - 3y = 0,$$

which does not have constant coefficients, nor is it Euler's equation.

Let's start with a simpler equation

$$y'' + y = 0.$$

Suppose

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + a_{n+1} x^{n+1} + \cdots .$$

Then

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

and

$$y'' = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots .$$

Since $y'' + y = 0$, we have

$$\begin{aligned} 0 = (2a_2 + a_0) + (3 \cdot 3a_3 + a_1)x + (4 \cdot 3a_4 + a_2)x^2 \\ + \cdots + ((n+2)(n+1)a_{n+2} + a_n)x^n + \cdots , \end{aligned}$$

which leads to

$$2a_2 + a_0 = 0, \qquad 3 \cdot 2a_3 + a_1 = 0, \qquad \text{etc.},$$

or

$$a_2 = -\frac{a_0}{2}, \qquad a_3 = -\frac{a_1}{3 \cdot 2}, \qquad a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2},$$

and

$$a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}.$$

Substituting in the equation for $y$,

$$\begin{aligned} y =& a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \cdots \\ &= a_0\Big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\Big) \\ &+ a_1\Big(x - \frac{x^3}{3!} + \cdots\Big). \end{aligned}$$

We conclude

$$y = a_0 \cos x + a_1 \sin x.$$

Note

$$y(0) = a_0, \qquad y'(0) = a_1.$$

Typically, one gets $a_0(\cdots) + a_1(\cdots)$, and one does not recognize the Taylor series. Nevertheless the Taylor series is good for calculations.

If we are interested in the behavior of $y$ near 3 instead of near 0, we use a Taylor series about 3 instead.

If our equation is, for example,

$$y'' - 3xy' + 8x^2 y = e^x,$$

we would also expand $e^x$ in a power series, and instead of setting the coefficients of $x^i$ to 0, we instead would set them equal to the corresponding coefficient in the Taylor series expansion of $e^x$.

Let's now go back to our original equation $y'' + xy' - 3y = 0$. The formulas for $y'$ and $y''$ are the same, and we can write

$$\begin{aligned}[2a_2 + 3{\cdot}2a_3x + 4\cdot 3a_4x^2 + \cdots]\\ &+ x[a_1 + 2a_2x + 3a_3x^2 + \cdots]\\ &- 3[a_0 + a_1x + a_2x^2 + \cdots] = 0.\end{aligned}$$

Looking at the constant coefficients and the coefficients of $x$, $x^2$, etc., we get a series of equations. The constant coefficient is

$$2a_2 - 3a_0 = 0,$$

which leads to

$$a_2 = \frac{3}{2}a_0.$$

The coefficient of $x$ is

$$3\cdot 2a_3 + a_1 - 3a_1 = 0,$$

which leads to

$$a_3 = \frac{1}{3}a_1.$$

The coefficient of $x^2$ is

$$4\cdot 3a_4 + 2a_2 - 3a_2 = 0,$$

or

$$a_4 = \frac{1}{12} a_2.$$

The coefficient of $x^3$ is

$$5 \cdot 4 a_5 = 0,$$

or $a_5 = 0$, and in fact $a_n = 0$ for all $n$ odd. It is not zero for $n$ even, and we can get a series of formulas for $a_6, a_8$, and so on.

One more example:

$$y'' + (x+1)y' + y = 0.$$

We write this as $y'' + xy' + y' + y = 0$. Then

$$\begin{aligned}[2a_2 + 3{\cdot}2a_3x + 4 \cdot 3a_4x^2 + \cdots]& \\ + x[a_1 + 2a_2x + 3a_3x^2 + \cdots]& \\ + [a_1 + 2a_2x + 3a_3x^2 + \cdots]& \\ + [a_0 + a_1x + a_2x^2 + \cdots] = 0.&\end{aligned}$$

This leads to the equations

$$2a_2 + a_1 + a_0 = 0, \qquad 6a_3 + a_1 + 2a_2 + a_1 = 0,$$

etc.

## 3.1 Regular singular points

Look at the more general equation

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

If $P(x_0) \neq 0$, then $x_0$ is an ordinary point, and the above theory works. If $P(x_0) = 0$, then $x_0$ is called a singular point.

If $x_0$ is a singular point and

$$\lim_{x \to x_0} \frac{xQ(x)}{P(x)} \qquad \text{and} \qquad \lim_{x \to x_0} \frac{x^2R(x)}{P(x)}$$

both exist, then $x_0$ is called a regular singular point.

Let's look at an example:

$$x^2y'' + xy' + (x - \tfrac{1}{9})y = 0.$$

If the $(x - \frac{1}{9})$ coefficient were instead $-\frac{1}{9}$, this would be the Euler equation with solution

$$c_1x^{1/3} + c_2x^{-1/3}.$$

Since neither $x^{1/3}$ nor $x^{-1/3}$ have Taylor series expansions about 0, a modification of the power series method is needed.

To handle the above differential equation near 0, we assume $y$ is of the form

$$y = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \cdots.$$

Then

$$y' = ra_0x^{r-1} + (r+1)a_1x^r + (r+2)a_2x^{r+1} + \cdots$$

and

$$y'' = r(r-1)a_0x^{r-2} + (r+1)ra_1x^{r-1} + (r+2)(r+1)a_2x^r + \cdots.$$

Substituting in the differential equation, we get

$$\begin{aligned} x^2y''+&xy' + (x - \tfrac{1}{9})y \\ &= [r(r-1)a_0 + ra_0 - \tfrac{1}{9}a_0]x^r \\ &+ [(r+1)ra_1 + (r+1)a_1 + a_0 - \tfrac{1}{9}a_1]x^{r+1} + \cdots. \end{aligned}$$

We set each of the coefficients equal to 0. If we don't want $a_0$ to be 0, we must have

$$r(r-1) + r - \tfrac{1}{9} = 0,$$

or $r = \frac{1}{3}, -\frac{1}{3}$. We get two solutions, one, say $y_1$, for $r = \frac{1}{3}$ and one, say $y_2$, for $r = -\frac{1}{3}$. The general solution is then $y = c_1y_1 + c_2y_2$.

To see how $y_1$ goes, from the coefficient of $x^{r+1}$ we get

$$(r+1)ra_1 + (r+1)a_1 + a_0 - \tfrac{1}{9}a_1 = 0,$$

or

$$\tfrac{4}{3} \cdot \tfrac{1}{3}a_1 + \tfrac{4}{3}a_1 + a_0 - \tfrac{1}{9}a_1 = 0$$

and we solve for $a_1$ in terms of $a_0$; it turns out $a_1 = -\frac{3}{5}a_0$. Using the coefficient of $x^{r+2}$, we get an equation that we can solve for $a_2$ in terms of

$a_1$, and hence in terms of $a_1$. We continue, getting all the $a_i$'s in terms of $a_0$. If we then substitute back in the formula for $y$, we get

$$y = a_0\Big(x^{1/3} - \tfrac{3}{5}x^{4/3} + \cdots\Big).$$

The expression inside the parentheses is $y_1$.

To get $y_2$, we do the same, but with $r = -\frac{1}{3}$. The general solution is then of the form

$$y = c_1 y_1 + c_2 y_2.$$

When one solves these equations, one gets out 2 values of $r$. If the two values of $r$ are not equal and do not differ by an integer, everything is fine. If they are equal or differ by an integer, one gets one solution using the larger value, but one has to work hard to get another.

As another example, look at Bessel's equation of order 0:

$$x^2y'' + xy' + x^2y = 0.$$

Substituting our formulas for $y, y'$, and $y''$ leads to

$$\begin{aligned} x^2[r(r-1)a_0x^{r-2} + (r+1)ra_1x^{r-1} + (r+2)(r+1)a_2x^r + \cdots] \\ + x[ra_0x^{r-1} + (r+1)a_1x^r + (r+2)a_2x^{r+1} + \cdots] \\ + x^2[a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \cdots] = 0. \end{aligned}$$

The coefficient of $x^r$, which is the smallest exponent, tells us that

$$r(r-1)a_0 + ra_0 = 0,$$

so either $a_0 = 0$, which leads to the uninteresting solution $y = 0$, or $r = 0$. Since there is only one value of $r$, we will only get one of the two solutions using this method.

The coefficient of $x^{r+1}$ is

$$r(r+1)a_1 + (r+1)a_1 = 0.$$

Since $r = 0$, this shows $a_1 = 0$.

From the coefficient of $x^{r+2}$ we get

$$(r+1)(r+2)a_2 + (r+2)a_2 + a_0 = 0,$$

or $4a_2 + a_0 = 0$, or $a_2 = -\frac{1}{4}a_0$.

We can continue to get as many coefficients as we want.

# 4 Boundary value problems

A boundary value problem is one like

$$y'' + 2y = 0, \qquad y(0) = 1, y(\pi) = 1$$

or

$$y'' + 2y = 0, \qquad y(0) = 0, y(\pi) = 0.$$

For the first, we get $r^2 + 2 = 0$, or $r = \pm\sqrt{2}i$, and so

$$y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x).$$

Since $y(0) = 1$, we get $1 = c_1$. Since $y(\pi) = 1$, we get

$$1 = \cos(\sqrt{2}\pi) + c_2 \sin(\sqrt{2}\pi),$$

and we can solve for $c_2$.

For the second, we proceed similarly. This time we get $c_1 = 0$ and

$$0 = c_2 \sin(\sqrt{2}\pi),$$

or $c_2 = 0$. Thus the only solution is $y = 0$.

Let us look at a third example,

$$y'' + 4y = 0, \qquad y(0) = 0, y(\pi) = 0.$$

This time the solution is $r = \pm 2i$, so

$$y = c_1 \cos(2x) + c_2 \sin(2x).$$

Again $y(0) = 0$ implies $c_1 = 0$. But putting $y(\pi) = 0$ gives

$$0 = y(\pi) = c_2 \sin(2\pi),$$

and $c_2$ can be anything. Therefore the solution is

$$y = c_2 \sin(2x).$$

We cannot determine $c_2$ because we have already specified two conditions $y(0) = 0$ and $y(\pi) = 0$.

Therefore some of these boundary value problems have no solutions and some have infinitely many solutions. This is similar to linear algebra. The pair of equations $x + y = 0$, $x + 2y = 0$ has only the zero solution. The pair of equations $x + y = 0$, $2x + 2y = 0$ has infinitely many solutions.

Let us look at the more general case

$$y'' + \lambda y = 0, \qquad y(0) = 0, y(L) = 0,$$

and see for what values there is a solution. We have $r^2 + \lambda = 0$, so $r = \pm\sqrt{\lambda}i$, so

$$y = c_1 \cos \lambda t + c_2 \sin \lambda t.$$

Since $y(0) = 0$, then $c_1 = 0$, and the solution is a multiple of $\sin(\sqrt{\lambda}t)$. In order for $y(L) = 0$, we must have $\sqrt{\lambda}L$ be a multiple of $\pi$, which says that

$$\lambda = \frac{n^2\pi^2}{L^2}.$$

The solution for this particular value of $\lambda$ is

$$y = c_2 \sin(\sqrt{\lambda}x) = c_2 \sin(n\pi x/L).$$

This is the analog of finding the eigenvalues and eigenvectors for a matrix.

# 5 Fourier series

Suppose $f$ is defined on an interval and suppose we can write

$$f(0) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \Big( a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \Big).$$

If this is the case, how do we find $a_m, b_m$?

We have $a_0/2$ instead of $a_0$ because this allows us to give a simpler expression for $a_0$. We get cosine terms, unlike the boundary value problems we looked at, because we are not saying that $f(0) = 0$. Finally, it turns out that if $f$ is piecewise differentiable, then the expansion is valid. $f$ does not need to be continuous. For example, we can let $f$ be a square function or a sawtooth function.

Recall

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B,$$

so

$$\cos A \cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)].$$

Then if $m, n$ are positive integers that are not equal,

$$\int_{-L}^{L} \cos(m\pi x/L)\cos(n\pi x/L)\,dx = \frac{1}{2}\Big[\int_{-L}^{L} \cos((m+n)\pi x/L)\,dx + \int_{-L}^{L} \cos((m-n)\pi x/L)\,dx\Big].$$

The first integral is equal to

$$\frac{1}{2}\Big[\frac{1}{(m+n)\pi/L}\sin((m+n)\pi x/L)\Big|_{-L}^{L}\Big] = 0$$

since $\sin(k\pi) = 0$ if $k$ is an integer. The second integral is also 0. We do the two cases when $m = n \neq 0$ and $m = n = 0$ similarly and obtain

$$\int_{-L}^{L} \cos(m\pi x/L)\cos(n\pi x/L)\,dx = \begin{cases} 0, & m \neq n; \\ L, & m = n \neq 0; \\ 2L, & m = n = 0. \end{cases}$$

If we use

$$\sin A \sin B = \cos(A+B) - \cos(A-B),$$

we get a similar result for the integral when the cosines are replaced by sines:

$$\int_{-L}^{L} \sin(m\pi x/L)\sin(n\pi x/L)\,dx = \begin{cases} 0, & m \neq n; \\ L, & m = n \neq 0; \\ 0, & m = n = 0. \end{cases}$$

If we start with

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B,$$

so

$$\sin A \cos B = \tfrac{1}{2}[\sin(A+B) + \sin(A-B)],$$

then we obtain

$$\int_{-L}^{L} \cos(m\pi x/L)\sin(n\pi x/L)\,dx = 0.$$

Therefore multiplying $f$ by $\cos\frac{n\pi x}{L}$ and integrating over $[-L, L]$ gives the $a_n$. To see this is more detail, we multiply term by term and get

$$\begin{aligned}
&\int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx \\
&\quad= \int_{-L}^{L} \frac{1}{2}a_0 \cos(n\pi x/L)\,dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos(m\pi x/L)\cos(n\pi x/L)\,dx \\
&\qquad + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin(m\pi x/L)\cos(n\pi x/L)\,dx.
\end{aligned}$$

All the integrals on the right are zero except for the one with $a_n$, where the integral equals $L$. Multiplying by $\sin(n\pi x/L)$ instead of cos gives $b_n$. Multiplying instead by the constant 1 gives $a_0$.

The formulas we get are

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx$$

and

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx.$$

For the first example, let

$$f(x) = \begin{cases} -1, & -1 \le x \le 0; \\ 1, & 0 < x \le 1. \end{cases}$$

This is a square wave. Here $L = 1$. $f$ is an odd function, and all the cosine terms are even, so $f(x)\cos(m\pi x/L)$ is an odd function. Hence its integral is 0, and we conclude all the $a_m$'s are equal to 0. To find $b_m$, we have

$$b_m = \int_{-1}^{1} f(x)\sin(m\pi x)\,dx = 2\int_{0}^{1} \sin(m\pi x)\,dx$$

since $f(x)\sin(m\pi x)$ is an even function. The last integral is

$$-2\frac{\cos(m\pi x)}{m\pi}\Big|_0^1 = -\frac{2}{m\pi}[\cos(m\pi) - 1].$$

$\cos(m\pi)$ equals $+1$ if $m$ is even and $-1$ if $m$ is odd. So

$$b_m = \begin{cases} 0, & m \text{ even;} \\ 4/(m\pi), & m \text{ odd.} \end{cases}$$

Substituting,

$$f(x) = \frac{4}{\pi}\sin(\pi x) + \frac{4}{3\pi}\sin(3\pi x) + \frac{4}{5\pi}\sin(5\pi x) + \cdots .$$

A second example is the sawtooth function:

$$f(x) = x, \qquad -1 \le x \le 1.$$

As before the $a_m$ are zero because $x$ is an odd function. We recall

$$x\cos Ax\,dx = \frac{1}{A}x\sin Ax + \frac{1}{A^2}\cos Ax + C,$$

$$x\sin Ax\,dx = -\frac{1}{A}x\cos Ax + \frac{1}{A^2}\sin Ax + C.$$

Then

$$\begin{aligned} b_m &= 2\int_0^1 x\sin(m\pi x)\,dx \\ &= -\frac{2}{m\pi}x\cos(m\pi x)\Big|_0^1 + \frac{2}{(m\pi)^2}\sin(m\pi x)\Big|_0^1 \\ &= \begin{cases} -2/m\pi, & m \text{ odd;} \\ 2/m\pi, & m \text{ even.} \end{cases} \end{aligned}$$

A third example is

$$f(x) = \begin{cases} -x & -2 \le x < 0 \\ x & 0 \le x < 2 \end{cases}.$$

Here $L = 2$ and we get $b_m = 0$, $a_0 = 2$, and

$$a_m = \begin{cases} -8/(m\pi)^2, & m \text{ odd;} \\ 0, & m \text{ even.} \end{cases}$$

## 5.1 Related information

From Euler's identities, $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$. Adding and dividing by 2 gives the formula for cosine, and subtracting and dividing by $2i$ gives the formula for sine:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

If we use this in the Fourier series expansion of a function, and collect terms, we get

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Note the sum goes over negative $n$'s as well as positive ones. This form of Fourier series is quite common.

Conversely, given a series in terms of sums of the $e^{inx}$, we can use Euler's formula to write it in terms of sines and cosines.

Comparing Fourier series with Taylor series, Fourier series are less sensitive to changes in the function $f$, they capture global behavior better, and they exist for more functions. Taylor series are easier to work with and they converge more rapidly.

Parseval's identity is a way of expressing $\int_{-L}^{L} f(x)^2 dx$ is terms of sums of the squares of the coefficients. This gives rise to some very pretty formulas, and also has theoretical importance, but not so much practical importance. To see how this goes,

$$\begin{aligned} f(x)^2 = \sum_{m,n} & a_m a_n \cos(m\pi x/L) \cos(n\pi x/L) \\ & + \sum_{m,n} a_m b_n \cos(m\pi x/L) \sin(n\pi x/L) \\ & + \sum_{m,n} b_m b_n \sin(m\pi x/L) \sin(n\pi x/L) \\ & + \tfrac{1}{4} a_0^2 + \sum_{m=1}^{\infty} \tfrac{1}{2} a_0 a_m \cos(m\pi x/L) \\ & + \sum_{m=1}^{\infty} \tfrac{1}{2} a_0 b_m \sin(m\pi x/L). \end{aligned}$$

If we integrate over $x$ from $-L$ to $L$, all the integrals are 0 except for the terms where $n = m$ in the first line and in the third line. Thus we get

$$\int_{-L}^{L} f(x)^2\,dx = \frac{1}{2}La_0^2 + L\sum_{m=1}^{\infty} a_m^2 + L\sum_{m=1}^{\infty} b_m^2.$$

Dividing by $L$ gives Parseval's identity:

$$\frac{1}{L}\int_{-L}^{L} f(x)^2\,dx = \frac{1}{2}a_0^2 + \sum_{m=1}^{\infty} a_m^2 + \sum_{m=1}^{\infty} b_m^2.$$

In the case of the function $f(x) = x$ on $-1, 1]$, all the $a_m$'s are equal to 0, and

$$\int_{-L}^{L} f(x)^2 = \frac{x^3}{3}\bigg|_{-1}^{1} = \frac{2}{3}.$$

Here Parseval's identity says that

$$\frac{2}{3} = \frac{4}{\pi^2} + \frac{4}{4\pi^2} + \frac{16}{9\pi^2} + \cdots,$$

or

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$

Recall from linear algebra that an orthonormal basis $\{v_1, \ldots, v_n\}$ is a collection of vectors such that every vector $v$ in $\mathbb{R}^n$ can be written as $v = \sum_{i=1}^{n} c_i v_i$ for some constants $c_i$, $v_i \cdot v_j = 0$ if $i \neq j$, and the inner product equals 1 if $i = j$. To find the $c_j$'s, we dot $v$ with $v_j$ to get

$$v \cdot v_j = c_1 v_1 \cdot v_j + \cdots + c_n v_n \cdot v_j = c_j.$$

This is what we are doing in Fourier series. Let $L = 1$ for simplicity, define $f \cdot g = \int_{-L}^{L} f(x)g(x)\,dx$, and let the $v_j$'s be $\cos(n\pi x/L)$'s and $\sin(n\pi x/L)$'s. The coefficients are given by taking the inner product of $f$ with the $v_j$'s.

Parseval's identity is just the Pythagorean theorem.

$f$ is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. When $f$ is even, we get a Fourier cosine series, and if $f$ is odd, a Fourier sine series.

If we have a function on $[0, L]$, we can extend it to be even on $[-L, L]$ to get a Fourier cosine series by setting $f(-x) = f(x)$, or odd to get a Fourier sine series by setting $f(-x) = -f(x)$. Note that if we do this, we get a function on $[-L, L]$, compared to one on $[0, L]$. The reason for doing one rather than the other has to do with applications. When using Fourier series to solve the heat equation where the ends of a thin rod are kept at $0^o$ $C$, one uses a Fourier sine series since $\sin(n\pi x/L)$ is equal to 0 when $x = 0$ and when $x = L$.

# 6 Heat equation

We consider a thin rod of uniform thickness and uniform material of length $L$ and want to figure out the temperature $u(x, t)$ at the point $x$ and time $t$. We assume that the left and right hand ends are kept at $0^o$ $C$, which means that $u(0, t) = u(L, t) = 0$ for all $t$. We assume that we are given an initial temperature distribution at time 0, so we are given $u(x, 0) = f(x)$ for some specified function $f$.

Let us look at the equation

$$\frac{\partial u}{\partial t} = K^2 \frac{\partial^2 u}{\partial x^2},$$

with boundary conditions

$$u(x, 0) = f(x)$$

for all $x$ and also

$$u(0, t) = 0, \qquad u(L, t) = 0.$$

This is a PDE (partial differential equation), called the heat equation. $K$ is related to the conductivity of the material. Let us assume the solution is of the form

$$u(x, t) = X(x)T(t),$$

where $X$ is a function only of $x$ and $T$ is a function only of $t$. Since we consider $x$ a constant when we take the partial derivatives with respect to $t$,

$$\frac{\partial u}{\partial t}(x, t) = X(x)T'(t),$$

and similarly for $\frac{\partial^2 u}{\partial x^2}$. We then have

$$K^2 X''(x)T(t) = X(x)T'(t),$$

or

$$\frac{X''}{X} = \frac{1}{K^2}\frac{T'}{T}.$$

The left hand side is a function only of $x$ and not of $t$ and the right hand side is a function only of $t$ and not of $x$. The only way that can happen is if they are equal to a constant, say $-\lambda$. So

$$X'' + \lambda X = 0$$

and

$$T' + K^2\lambda T = 0.$$

The boundary values translate to $X(0) = X(L) = 0$.

When we look at the first equation, we see that we must have $\lambda > 0$. We can't have $\lambda = 0$, because then the solution would be $X(x) = c_1 + c_2 x$, and the boundary conditions lead to $c_1 = c_2 = 0$, which is uninteresting. We can't have $\lambda < 0$, or we would have $X(x) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{-\lambda}t}$, and again the boundary conditions would imply that $c_1 = c_2 = 0$.

The solutions to the first equation are

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

Putting in the boundary conditions, we find that $c_1 = 0$ and also $c_2 = 0$ unless $\sqrt{\lambda}$ is an integer multiple of $\pi$. When it is, we get the solution

$$X = c_n \sin\frac{n\pi x}{L}, \qquad \lambda = \frac{n^2\pi^2}{L^2},$$

where $c_n$ is any constant and then

$$T = e^{-n^2\pi^2K^2t/L^2}.$$

So

$$u_n = c_n e^{-n^2\pi^2K^2t/L^2} \sin\frac{n\pi x}{L}$$

solves the PDE for any constant $c_n$ and any integer $n$. Because $\sin 0 = 0$ and $\sin(-x) = -\sin x$, we may restrict attention to positive integers.

By linearity,

$$u = \sum_n c_n u_n$$

is a solution. Since $u(x,0) = f(x)$, we have

$$f(x) = \sum_n c_n \sin \frac{n\pi x}{L}.$$

So we find our solution as follows.

(1) We extend $f$ to $[-L, L]$ by letting $f(-x) = -f(x)$.
(2) Such an $f$ is odd, and its Fourier series will be a Fourier sine series.
(3) We determine the coefficients $c_n$ by expanding $f$ in a Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L).$$

(4) We then have our solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L) e^{-K^2 n^2 \pi^2 t/L^2}.$$

For example, suppose $L = 1$, $K = 1$, and

$$f(x) = 2\sin(\pi x) - 3\sin(3\pi x).$$

This is already a Fourier sine series with $b_1 = 2$ and $b_3 = -3$, and all the other $b_m$'s are zero. The solution to the heat equation is then

$$u(x,t) = 2\sin(\pi x)e^{-\pi^2 t} - 3\sin(3\pi x)e^{-9\pi^2 t}.$$

For another example, let $K$=2 and $L = 1$, and suppose $f(x) = x$. To make this an odd function, we extend the definition of $f$ to $[-1, 1]$ by setting $f(-x) = -f(x)$. We did the Fourier series for this one already and found that $b_m = -2/m\pi$ if $m$ is odd and $b_m = 2/m\pi$ if $m$ is even. Then the solution to the heat equation is

$$f(x) = -\frac{2}{\pi}\sin(\pi x)e^{-4\pi t} + \frac{2}{2\pi}\sin(2\pi x)e^{-16\pi t} - \frac{2}{3\pi}\sin(3\pi x)e^{-36\pi t} + \cdots .$$

# 7 Other boundary conditions

If we have the boundary conditions

$$u(0,t) = T_1, \qquad u(L,t) = T_2,$$

and $u(x,t) = f(x)$, look at the heat equation for $\widetilde{u}$ with boundary conditions $\widetilde{u}(0,t) = \widetilde{u}(L,t) = 0$,

$$\widetilde{u}(x,0) = f(x) - [(T_2 - T_1)\frac{x}{L} + T_1].$$

The solution will then be

$$u = \widetilde{u} + (T_2 - T_1)\frac{x}{L} + T_1.$$

We can check that $u$ satisfies the heat equation with the given initial and boundary conditions.

If we have insulated ends:

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0,$$

where $u_x = \partial u/\partial x$, we proceed as above, but get

$$X = c_1 \sin \lambda x + c_2 \cos \lambda x.$$

From the boundary values, we get $c_1 = 0$ and

$$u(x,t) = \frac{c_0}{2} + \sum c_n e^{-n^2\pi^2K^2t/L^2} \cos \frac{n\pi x}{L}.$$

Given an initial condition $u(x,0) = f(x)$, we extend $f$ to $[-L,L]$ so that $f$ is even, which means that its Fourier series will only have cosine terms, and then we use the Fourier series to determine the $c_m$'s.

More general boundary value condition are things like

$$u(0,t) = 0, u_x(L,t) = 0$$

or $u_x(0,t) - h_1 u(0,t) = 0$.

# 8 Wave equation

The wave equation is

$$K^2 u_{xx} = u_{tt}$$

with boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0,$$

and

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x),$$

where $f(0) = f(L) = 0$ and $g(0) = g(L) = 0$.

We first assume $g$ is equal to 0 for all $x$. Supposing $u = XT$, we have

$$\frac{X''}{X} = \frac{1}{K^2}\frac{T''}{T} = -\lambda,$$

which gives us

$$X'' + \lambda X = 0$$

as before and

$$T'' + K^2 \lambda T = 0,$$

so that

$$T = k_1 \cos \frac{n\pi K t}{L} + k_2 \sin \frac{n\pi K t}{L}.$$

Since $u_t(x,0) = 0$, then $k_2$ must be 0.

Then

$$u = \sum c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi K t}{L},$$

where

$$f(x) = u(x,0) = \sum c_n \sin \frac{n\pi x}{L}.$$

Thus to solve the wave equation with the initial velocity equal to zero and initial position equal to $f$, first extend $f$ from $[0,L]$ to $[-L,L]$ so that it is odd (by setting $f(-x) = -f(x)$), finding the Fourier coefficients in the Fourier sine series, and then using those in the above equation.

An example: Suppose $L = 1$, $K^2 = 4$, and $f(x) = 3\sin(2\pi x) + 4\sin(3\pi x)$. Here $f$ is already odd, so the extension of $f$ to $[-1,1]$ is $f$ itself. The Fourier coefficents are $b_2 = 3$ and $b_3 = 4$ and the rest of the $b_m$'s are zero. The solution to the wave equation is then

$$u(x,t) = 3\sin(2\pi x)\cos(4\pi t) + 4\sin(3\pi x)\cos(6\pi t).$$

Next we assume $f$ is identically 0, and similarly get

$$u = \sum k_n \sin\frac{n\pi x}{L}\sin\frac{n\pi Kt}{L}.$$

We differentiate with respect to $t$, set $t = 0$, and use Fourier series to get $k_n$.

In the case where neither $f$ nor $g$ is identically 0, we do $f$ and $g$ separately (set $g = 0$, then set $f = 0$) and add.

# 9 Laplace equation

The Laplace equation is

$$u_{xx} + u_{yy} = 0.$$

We suppose we are in a square $[0,a] \times [0,b]$ with boundary conditions 0 on the top, left, and bottom, and $u(a,y) = f(y)$ on the right.

We write

$$u = XY,$$

which leads to

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

The boundary conditions become $X(0) = 0$, $Y(0) = Y(b) = 0$. We get

$$Y = \sin\frac{n\pi y}{b},$$

and

$$X = c_1 e^{n\pi x/b} + c_2 e^{-n\pi x/b}.$$

The boundary condition $X(0) = 0$ implies $c_1 = -c_2$, or

$$X = c_1 \sinh\frac{n\pi x}{b}.$$

Our solution becomes

$$u = \sum c_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}.$$

If we put in $x = a$ and let

$$b_n = c_n \sinh \frac{n\pi a}{b},$$

then

$$f(y) = u(a, y) = \sum_n b_n \sin \frac{n\pi y}{b}.$$

We can also look at the Laplace equation in a circle of radius 1. In this case we make a change of variables and get

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

with boundary condition

$$u(1, \theta) = f(\theta).$$

We write

$$u = R\Theta$$

and get

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0,$$

which leads to

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

If $\lambda < 0$, then

$$\Theta = c_1 e^{a_1 \theta} + c_2 e^{a_2 \theta},$$

which is not periodic. If $\lambda = 0$, we have

$$\Theta = c_1 + c_2 \theta.$$

To be periodic, we have to have $c_2 = 0$. But then

$$r^2 R'' + rR' = 0$$

implies

$$R = k_1 + k_2 \ln r,$$

and either $u$ is not bounded, or else $u$ is constant.

So to get anything interesting, we need $\lambda > 0$. Let $\mu = \sqrt{\lambda}$, so that $\lambda = \mu^2$. We get

$$\Theta = c_1 \sin \mu\theta + c_2 \cos \mu\theta$$

and

$$r^2 R'' + rR' - \mu^2 R = 0,$$

which implies

$$R = k_1 r^{\mu} + k_2 r^{-\mu}.$$

To be periodic, we must have $\mu = n$. To keep $u$ bounded, we must have $k_2 = 0$. So our solution is

$$u(r, \theta) = \frac{c_0}{2} + \sum r^n (c_n \cos n\theta + k_n \sin n\theta).$$

Putting in $r = 1$, we expand $f(\theta)$ in a Fourier series to get the coefficients $c_n, k_n$.

# 10 Sturm-Liouville theory

One might want to solve more general PDEs, such as

$$r(x)u_t = [p(x)u_x]_x - q(x)u + F(x, t),$$

with general boundary conditions such as

$$u_x(0, t) - h_1 u(0, t) = 0, \qquad u_x(L, t) - h_2 u(L, t) = 0.$$

One could also look at more general regions.

Let us start by looking at the boundary value problem

$$(p(x)y'(x))' - q(x)y(x) = -\lambda r(x)y(x), \tag{10.1}$$

with $y(0) = y(L) = 0$. Using the product rule, this can also be written as

$$p(x)y'' + p'(x)y' - q(x)y = -\lambda r(x)y.$$

There will be solutions for some $\lambda$'s and no solution for others. In the particular case when $p(x) = 1$ for all $x$, $q(x) = 0$ for all $x$, and $r(x) = 1$ for all $x$, we are led to the equation

$$y'' + \lambda y = 0, \qquad y(0) = y(L) = 0,$$

where we know there is a solution if $\lambda = n^2\pi^2/L^2$ and not otherwise, and the solution is $y(x) = c\sin(n\pi x/L)$.

The equation (10.1) is more general than it appears at first. If we have the equation

$$A(x)y'' + B(x)y' + C(x)y = -\lambda D(x)y,$$

we can divide through by $A$ to get

$$y'' + \frac{B}{A}y' + \frac{C}{A}y = -\lambda\frac{D}{A}y,$$

and then multiply by $p$, where

$$p(x) = e^{\int B(t)/A(t)\,dt}.$$

Then

$$p' = p\frac{B}{A},$$

using the fundamental theorem of calculus, and we have

$$py'' + p'y + p\frac{C}{A}y = -\lambda p\frac{D}{A}y.$$

Setting $q = -pC/A$ and $r = -pD/A$ puts the equation in the form of (10.1).

Suppose we set

$$\mathcal{L}f(x) = -(pf')' - qf.$$

Our equation then becomes

$$\mathcal{L}y = \lambda ry.$$

We are looking for eigenvalues $\lambda$ of the operator $\mathcal{L}$ and looking for the corresponding eigenfuntions. This is completely analogous to finding the eigenvalues and eigenvectors of a matrix $A$.

For the set of eigenvectors to be a basis, it is necessary for $A$ to be symmetric. Similarly, we want some sort of symmetry for $\mathcal{L}$. The operator $\mathcal{L}$ is symmetric in the following sense:

$$\int f(x)(\mathcal{L}g)(x)\,dx = \int (\mathcal{L}f)(x)g(x)\,dx.$$

We need some boundary conditions and we impose the ones $y(0) = y(L) = 0$ and we suppose $f$ and $g$ both satisfy these conditions.

To show that $\mathcal{L}$ is symmetric, let us do the case where $\mathcal{L}y = y''$. We have

$$\int_0^L f(\mathcal{L}g)\,dx = \int_0^L fg'' = fg'\Big|_0^L - \int_0^L f'g',$$

using integration by parts. Since $f$ satisfies the boundary conditions, the first term is 0, and we get

$$\int_0^L f'g'.$$

Integrating by parts again gives us

$$\int_0^L (\mathcal{L}f)g.$$

If we do the general case, but where $q = 0$, we have

$$\int_0^L f(pg')' = -\int_0^L f'pg'$$

by integration by parts, and integrating by parts a second time gives

$$\int_0^L g(pf')'.$$

In the general case, we will get a sequence of eigenvalues $\lambda_1, \lambda_2, \ldots$ with corresponding eigenfunctions $y_1, y_2, \ldots$, so that

$$\mathcal{L}y_m = \lambda_m y_m.$$

The symmetry of $\mathcal{L}$ implies that $\int r y_m y_n = 0$ if $m \neq n$. To see this, if $\lambda_m \neq \lambda_n$, we have

$$\begin{aligned}\lambda_m \int r y_m y_n &= \int \lambda_m r y_m y_n = \int (\mathcal{L} y_m) y_n \\ &= \int y_m (\mathcal{L} y_n) = \int y_m \lambda_n r y_n \\ &= \lambda_n \int r y_m y_n.\end{aligned}$$

This is only possible if the integral is zero.

Once we have this, we can write a given function $f$ as

$$f = c_1 y_1 + c_2 y_2 + \cdots .$$

To identify the $c_j$'s, multiply both sides by $r y_j$ and integrate to get

$$\begin{aligned}\int_0^L r f y_j &= c_1 \int_0^L r y_1 y_j + c_2 \int_0^L r y_2 y_j + \cdots \\ &= c_j \int_0^L r y_j^2,\end{aligned}$$

and hence

$$c_j = \frac{\int_0^L r(x) f(x) y_j(x)\, dx}{\int_0^L r(x) y_j(x)^2\, dx}.$$

As an example, look at

$$y'' + \lambda y = 0, \qquad y'(0) = 0, y'(1) + y(1) = 0.$$

The boundary conditions are not of the form $Y(0) = 0$, $y(L) = 0$, but the same argument as before shows that $\mathcal{L}$ is symmetric and there are eigenvalues and eigenfunctions.

The general solution is, as before,

$$y = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x).$$

Then

$$y' = -c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x).$$

Putting in $y'(0) = 0$ leads to $c_2 = 0$ (or else $\lambda = 0$, which leads to $y = 0$, which is not interesting). Putting in $y'(1) + y(1) = 0$ leads to

$$-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}) + c_1\cos(\sqrt{\lambda}) = 0.$$

Therefore the equation the eigenvalues must satisfy is

$$\frac{1}{\sqrt{\lambda}} = \tan\sqrt{\lambda}.$$

We can see that there are infinitely many solutions graphically.

## 10.1 Nonhomogeneous equations

Suppose we are looking at

$$Ly = \mu r y + f.$$

Let $y_n$ be the normalized eigenfunctions, which means

$$\int_0^1 y_n^2 r = 1.$$

We assume $y = \sum_n b_n y_n$. Suppose $f/r = \sum c_n y_n$. Then

$$Ly = \sum b_n L y_n = \sum_n b_n \lambda_n r y_n,$$

and we get

$$\sum b_n \lambda_n r y_n = \sum \mu r_n b_n y_n + \sum c_n r y_n.$$

So

$$\sum [b_n(\lambda_n - \mu) - r_n] y_n = 0,$$

which leads to

$$b_n = \frac{c_n}{\lambda_n - \mu}$$

and

$$y = \sum \frac{c_n}{\lambda_n - \mu} y_n.$$

As an example, consider

$$y'' + 2y = \sin 3\pi x - 4\sin 5\pi x,$$

with boundary conditions $y(0) = 0, y(1) = 0$. The first thing we do is rewrite the equation in the form:

$$-py'' - p'y' + qy = \mu ry + f,$$

or

$$py'' + p'y' - qy + \mu ry + f = 0.$$

Our equation fits into this form if we let $p = 1$, $q = 0$, $r = 1$, $\mu = 2$, and $f = -\sin 3\pi x + 4\sin 5\pi x$. We find the eigenfunctions are $\sin n\pi x$ with corresponding eigenvalues $n^2\pi^2$. So the normalized eigenfunctions are

$$y_n = \sqrt{2}\sin n\pi x.$$

We expand $f$ as $\sum c_n y_n$, and taking into account the normalization,

$$c_3 = -\sqrt{2}/2, \qquad c_5 = 2\sqrt{2},$$

and all the other $c_n$ are 0. So

$$b_3 = -\frac{\sqrt{2}/2}{9\pi^2 - 2}, \qquad b_5 = \frac{2\sqrt{2}}{25\pi^2 - 2},$$

and all the other $b_n$ are 0. Hence the solution is

$$y = -\frac{\sqrt{2}/2}{9\pi^2 - 2}\sqrt{2}\sin 3\pi x + \frac{2\sqrt{2}}{25\pi^2 - 2}\sqrt{2}\sin 5\pi x.$$

## 10.2 PDEs with boundary conditions

If one has the PDE

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x_2} + B\frac{\partial u}{\partial x} + Cu,$$

where $A$, $B$, and $C$ are functions of $x$, one first divides by $A$ and then multiplies by $p = e^{\int B/A}$ to get it into the form

$$ru_t = [pu_x]_x - qu.$$

Next we try $u = XT$ to see if we can get some solutions that will be the building blocks for the general solution. We get

$$rXT' = (pX')'T - qXT,$$

or

$$\frac{(pX')'}{rX} - \frac{q}{r} = \frac{T'}{T} = -\lambda,$$

and one is led to solving

$$(py')' - qy + \lambda r y = 0.$$

If, for example, one has the boundary conditions $u(0,t) = u(L,t)$ for all $t$, this leads to $X(0) = X(L) = 0$, although other boundary conditions are possible.

We apply the Sturm-Liouville theory and get a sequence $\lambda_1 \leq \lambda_2 \leq \cdots$ and functions $y_1, y_2, \ldots$ such that a given function $f$ can be written as

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x),$$

with

$$c_n = \frac{\int_0^L r(x) f(x) y_n(x)\, dx}{\int_0^L r(x) y_n(x)^2\, dx}. \tag{10.2}$$

The solution to the equation

$$\frac{T'}{T} = -\lambda$$

is

$$T = e^{-\lambda t},$$

so the general solution to the PDE is

$$u(x,t) = \sum_{n=1}^{\infty} c_n y_n(x) e^{-\lambda_n t}. \tag{10.3}$$

To determine the $c_n$, we use a given initial value condition $u(x,0) = f(x)$ for all $x$ to set $t = 0$ in (10.3) and obtain

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n y_n(x).$$

We therefore use the given $f$ and (10.2) to determine the $c_n$'s and then substitute them into (10.3) to find the final solution to the problem.

# 11 Vibrating membrane

Let's look at a problem that puts together serveral of the things we've learned. We look at a vibrating circular membrane tied down at the edge, for example, a drum head.

The equation it solves is the wave equation:

$$u_{tt} = a^2(u_{xx} + u_{yy}).$$

It has the boundary condition $u = 0$ on the boundary of the disk. We assume that an initial displacement is given, but that the initial velocity is 0. We also assume that the initial displacement is radially symmetric, that is, it depends only on $r$ and not $\theta$, if we consider polar coordinates.

If we change to polar coordinates, we get

$$u_{tt} = a^2\Big(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\Big).$$

At $r = 1$, $u = 0$. Since our initial condition does not depend on $\theta$, it is very reasonable that our solution won't either. Thus we drop the $\theta$ and consider $u(r, t)$. We have $u(1, t) = 0$ and $u(r, 0) = f(r)$.

We try separation of variables, and look for solutions of the form

$$u(r, t) = R(r)T(t).$$

We then get

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{1}{a^2} \cdot \frac{T''}{T} = -\lambda^2.$$

We then have the two ordinary differential equations

$$r^2R'' + rR' + \lambda^2 r^2 R = 0$$

and

$$T'' + \lambda^2 a^2 T = 0.$$

The first one has boundary conditions and will only have a solution for some $\lambda$. Once we find $\lambda$, we solve the second and obtain

$$T(t) = k_1 \sin(\lambda at) + k_2 \cos(\lambda at).$$

If we let $s = \lambda r$, the first equation becomes

$$s^2 R'' + sR' + s^2 R = 0,$$

which is Bessel's equation. We do our standard method of solving this: assume

$$y = a_0 x^r + a_1 x^{r+1} + \cdots .$$

The $r$ here is not the same as the one in our equation, but we will be using this $r$ for just a moment and then drop it. Our indicial equation is

$$a_0 r(r-1) + a_0 r = 0,$$

which implies $r = 0$. We have

$$a_0[r(r-1)+r]x^r + a_1[(r+1)r + (r+1)]x^{r+1} \\ + \sum_{n=2}^{\infty}[a_n(r+n)(r+n-1) + (r+n)a_{n-2}]x^{r+n}.$$

Since $r = 0$, we get that $a_1 = 0$. We also have the general recursion relationship

$$a_n = -\frac{a_{n-2}}{(r+n)(r+n-1)+(r+n)} = -\frac{a_{n-2}}{n(n-1)+n} = -\frac{a_{n-2}}{n^2}.$$

So $a_n = 0$ for $n$ odd and $a_2 = -a_0/4$, $a_4 = a_0/64$, and in general

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m!)^2}.$$

Therefore one solution is

$$J_0(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} x^{2m}.$$

This is called the Bessel function of the first kind of order 0.

Other techniques are needed to find the other solution, but it turns out to be of the form

$$Y_0(x) = J_0(x)\ln x + \text{ a nice Taylor series.}$$

(The $Y_0$ is also standard notation.)

So we get

$$R(r) = c_1 J_0(s) + c_2 Y_0(s) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r).$$

In our boundary value problem, we didn't specify $R(0)$, but we do know it is finite, while $Y_0(0)$ is infinite. Hence $c_2$ is 0. Thus our solution is

$$R(r) = c_1 J_0(\lambda r).$$

Since $R(1) = 0$, we need to choose a value of $\lambda$ such that $J_0(\lambda) = 0$.

It turns out that $J_0$ oscillates infinitely often as $r \to \infty$. This isn't completely surprising. When $s$ is large the equation

$$s^2 R'' + sR' + s^2 R = 0$$

looks a lot like $s^2 R'' + s^2 R = 0$, which has solutions cos and sin. It turns out that the zeros of $J_0$ are approximately (but not exactly) equal to $(n - \frac{1}{4})\pi$.

What we have so far is that a solution to our PDE satisfying the boundary condition on the boundary of the disk is

$$u(r, t) = c_n J_0(\lambda_n r) \sin(\lambda_n at) + d_n J_0(\lambda_n r) \cos(\lambda_n at).$$

Therefore the general solution is

$$u(r, t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) \sin(\lambda_n at) + d_n J_0(\lambda_n r) \cos(\lambda_n at).$$

Because

$$u_t(r, 0) = \sum \lambda_n a c_n J_0(\lambda_n r) = 0,$$

we can see that the $c_n$ are zero. Therefore the solution is

$$u(r, t) = \sum_{n=1}^{\infty} d_n J_0(\lambda_n r) \cos(\lambda_n at).$$

Since $u(r, 0) = f(r)$, we have

$$d_n = \frac{\int_0^1 r f(r) J_0(\lambda_n r)\, dr}{\int_0^1 r J_0(\lambda_n r)^2\, dr}.$$

## 12 Linear systems

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $A$ a $2 \times 2$ matrix, and suppose we want to solve

$$X' = AX.$$

We assume the solution is of the form $X = We^{rt}$, where $W$ is a $2 \times 1$ matrix. Then

$$X' = rWe^{rt},$$

so

$$rWe^{rt} = AWe^{rt},$$

or

$$AW = rW.$$

Thus $r$ are the eigenvalues of $A$ and $W$ are the corresponding eigenvectors. The solution becomes

$$X = c_1 W_1 e^{r_1 t} + c_2 W_2 e^{r_2 t}.$$

For example, suppose we have

$$x' = 3x + 2y,$$
$$y' = x + 2y,$$

where $x$ and $y$ are functions of time. We write this as

$$X' = AX,$$

where

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}.$$

The determinant of $A - rI$ is $r^2 - 5r + 4$, so the eigenvalues are 4 and 1. The corresponding eigenvectors are

$$W_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad W_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

This is fine is $r_1 \neq r_2$ are real. If $r = a \pm bi$, then we get

$$\begin{aligned} X &= W_1 e^{at} \cos bt + i W_1 e^{at} \sin bt \\ &\quad + W_2 e^{at} \cos bt - i W_2 e^{at} \sin bt \\ &= Z_1 e^{at} \cos bt + Z_2 e^{at} \sin bt. \end{aligned}$$

An example is where $X' = AX$ and

$$A = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}.$$

Then $r = -\frac{1}{2} \pm i$,

$$X_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-\frac{1}{2}+i} t,$$

and $X_2$ is similar. We get the two solutions by looking at the real and imaginary parts of $X_1$.

When we have repeated roots, one solution will be $X = W_1 e^{rt}$. For the other, we try

$$X = Z_1 t e^{rt} + Z_2 e^{rt}.$$

Then

$$X' = Z_1 e^{rt} + Z_1 r t e^{rt} + r Z_2 e^{rt},$$

and $X' = AX$ implies

$$Z_2 r e^{rt} + Z_1 e^{rt} + Z_1 r t e^{rt} = A Z_1 t e^{rt} + A Z_2 e^{rt},$$

or

$$r Z_1 = A Z_1$$

and

$$Z_1 + r Z_2 = A Z_2.$$

For $Z_1$ we get what we had before. We then solve $Z_1 + rZ_2 = AZ_2$. For example, if

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix},$$

then $r = 2$ and $Z_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}.$$

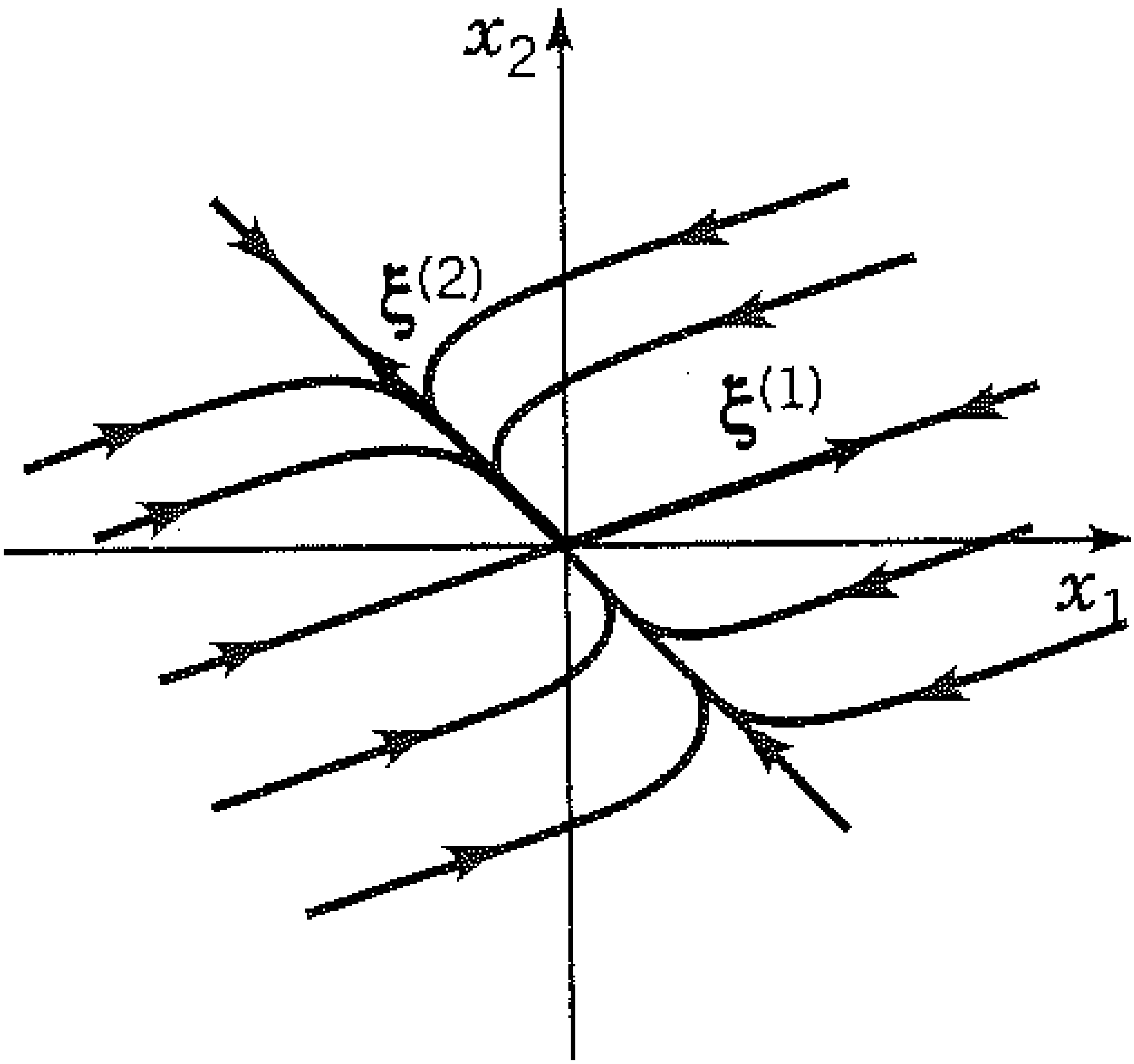
# 13 Nonlinear systems

## 13.1 Phase plane

If $r_1 < r_2$ and both are negative, we get the figure in the next page. Our solution is

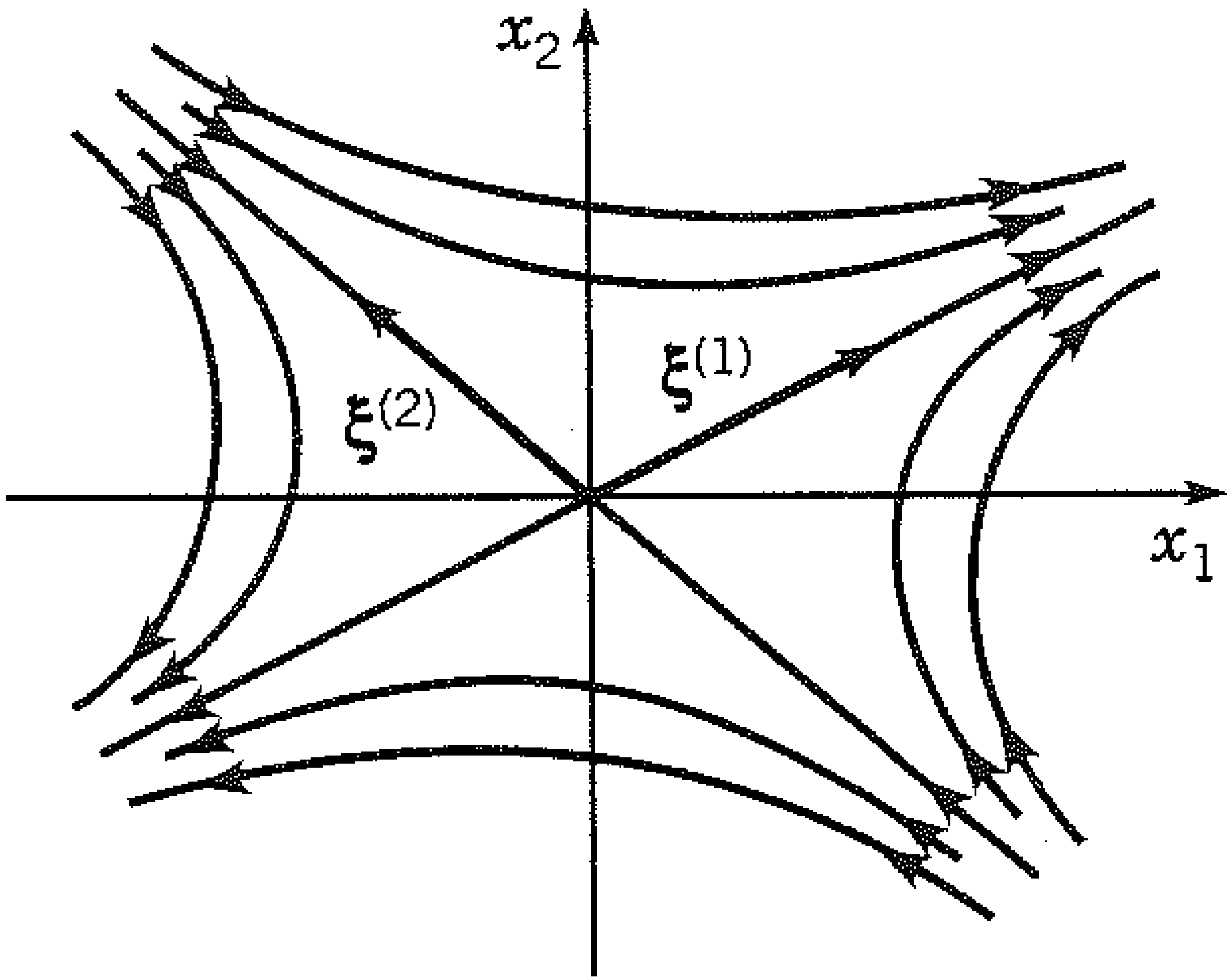
$$W_1 e^{-r_1 t} + W_2 e^{-r_2 t} = e^{-r_1 t}(W_1 + W_2 e^{-(r_2 - r_1)t}).$$

Here 0 is a stable equilibrium point We get a similar picture if both are positive.



$x_2$
$x_1$
$\xi^{(2)}$
$\xi^{(1)}$

If one of $r_1$ is positive and $r_2$ is negative, we get something like the next figure. If $x(0) = y(0) = 0$, then $x'(t) = y'(t) = 0$ for all $t$, and so 0 is an equilibrium point, but it is an unstable one. If we start just a bit away from 0, we get pushed further away.

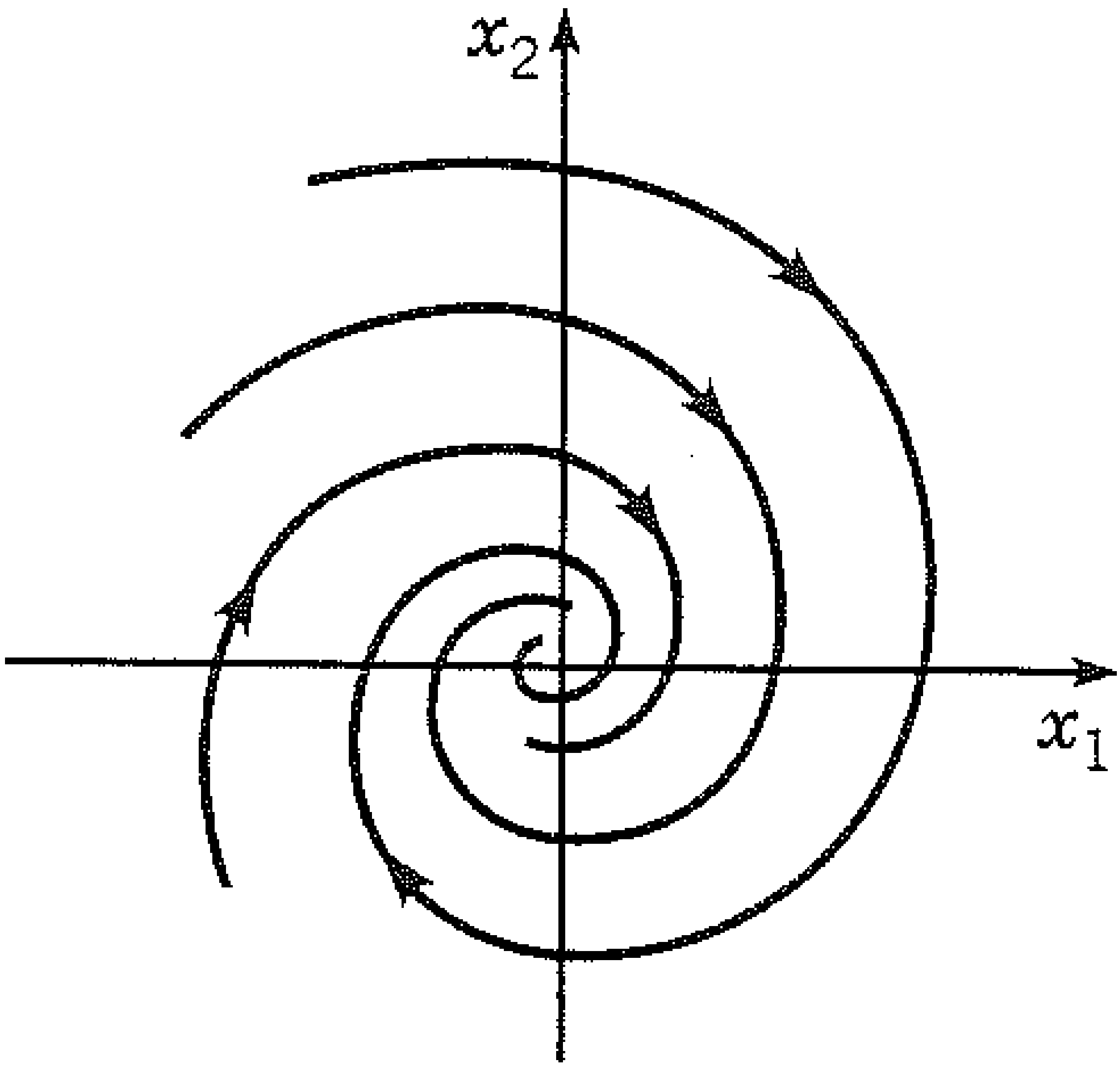


$x_2$
$\xi^{(1)}$
$\xi^{(2)}$
$x_1$

If the $r$'s are complex, our solution is of the form

$$W_1 e^{at} \cos bt + W_2 e^{at} \sin bt$$

with $a > 0$, the picture looks like the next page. If $a < 0$, the arrows point inward.


$x_2$
$x_1$

## 13.2 Autonomous equations

We'll look at

$$x' = F(x, y); \qquad y' = G(x, y).$$

Equations where there is no dependence on $t$ on the right hand side are called autonomous equations.

Let's start by looking at an example:

$$\begin{aligned} x' &= 4x + 2y + y^2; \\ y' &= 2x - y + x^2. \end{aligned}$$

We compare this to

$$\begin{aligned} x' &= 4x + 2y \\ y' &= 2x - y. \end{aligned}$$

When $x, y = 0$, we get $x', y' = 0$, so $(0, 0)$ is a critical point. When we solve this linear system, we get

$$r = \frac{3 \pm \sqrt{41}}{2}.$$

So one value of $r$ is positive and one negative, and $(0, 0)$ is an unstable equilibrium. Now when $x$ and $y$ are small, the $x^2$ and $y^2$ are negligible, so this also has an unstable equilibrium.

Another example is

$$\begin{aligned} x' &= x + y + 1 + y^2 \\ y' &= x - y + 4. \end{aligned}$$

We solve $y^2 + x + y + 1 = 0$, $x - y + 4 = 0$ to find the critical point. We end up with $(-2, 2)$ and $(-3, 1)$. For the second one, we do a transformation $u = x + 3$, $v = y - 1$, and our equation becomes

$$\begin{aligned} u' &= (u - 3) + (v + 1) + 1 + (v + 1)^2 \\ &= u + v + v^2 + 2v \\ v' &= (u - 3) - (v_1) + 4 = u - v. \end{aligned}$$

We now look at the linear approximation to see the behavior near these critical points.

## 13.3 Competing species

Here is an example.

$$x' = x(1 - x - y)$$
$$y' = y(2 - y - 3x).$$

To interpret this, $x'$ is approximately $x$ when there is plenty of food. It is approximately $1 - x$ when $x$ gets near the max, $1 - x - y$ when both do.

The critical points are $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(0, 2)$, and $(1, 0)$..

Another example:

$$x' = x(1 - x - y)$$
$$y' = y(2 - 2x - y).$$

$(1, 0)$ is a nontrivial equilibrium point.

Also:

$$x' = x(1 - x - y)$$
$$y' = y(\tfrac{3}{4} - y - \tfrac{1}{2}x),$$

wher $(\frac{1}{2}, \frac{1}{2})$ is the equilibrium point.

## 13.4 Predator-prey

Let $x$ be the prey, $y$ the predator.

$$x' = x(1 - 2y)$$
$$y' = y(-1 + 4).$$

The way to interpret this is $x' = x - 2xy$, and the $2xy$ is the rate that the prey is killed off, proportional to the number of encounters.

Another example:

$$x' = x(1 - \tfrac{1}{2}y)$$
$$y' = y(-\tfrac{3}{4} + \tfrac{1}{4}x).$$

The pattern is elliptical. Two properties: the period is independent of the initial condition, and the prey leads the predator by a quarter cycle.

# 14 Numerical solutions

## 14.1 Euler's method

$y' = f(t, y)$, with step size $h$. We start with $t_0, y_0$.

$$\begin{aligned} d_0 &= y_0' = f(t_0, y_0) \\ t_1 &= t_0 + h \\ y_1 &= y_0 + d_0 h \\ d_1 &= f(t_1, y_1) \\ t_2 &= t_1 + h \\ y_2 &= y_0 + d_1 h \end{aligned}$$

and so on.

An example.

The analyze the error: let $y = \phi(t)$ be the solution. By Taylor's expansion,

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \phi''(t_n)\frac{h^2}{2} + \cdots.$$

We have $y_{n+1} = \phi(t_n + h)$, $y_n = \phi(t_n)$, and $d_n h = \phi'(t_n)h$. So the error is of order

$$\|\phi''\|\frac{h^2}{2}.$$

There are two types of error: truncation error and round-off error.

## 14.2 Improved Euler method

The following equation is exact:

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y)\, dt.$$

Euler's method is

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n).$$

Better would be

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2} h,$$

but $f(t_{n+1}, y_{n+1})$ is unknown. The improved Euler method uses

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_{n+1}, y_n + hd_n))}{2} h.$$

### 14.3 Runge-Kutta

The error for Euler is $h^2$, the error for improved Euler is $h^3$, and for Runge-Kutta $h^5$. The formula for Runge-Kutta is

$$y_{n+1} = y_n + h\Big(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6}\Big),$$

where

$$\begin{aligned} k_{n1} &= f(t_n, y_n) \\ k_{n2} &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_{n1}) \\ k_{n3} &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_{n2}) \\ k_{n4} &= f(t_n + h, y_n + hk_{n3}). \end{aligned}$$

## 15 More numerical methods

The Runge-Kutta method is good for first order differential equations. What about second order equations?

### 15.1 Initial value problems

If we have

$$y'' = f(x, y, y'),$$

we know how to change this to a system of equations by setting $v = y'$ and then we have the equations

$$\begin{aligned} y' &= v; \\ v' &= f(x, y, v). \end{aligned}$$

For example, if we have the equation

$$y'' + 3xy' + yy' = x^2,$$

we can write this as

$$\begin{aligned} y' &= v; \\ v' + 3xv + yv &= x^2. \end{aligned}$$

Runge-Kutta works fine on systems of first order equations.

## 15.2 Boundary value problems

There are three methods, the shooting method, the finite difference method, and the Rayleigh-Ritz method, also known as the finite element method.

First the shooting method. Suppose we want to solve

$$y'' + 3xy' + yy' - x^2 = 0, \qquad y(0) = 1, \quad y(1) = 3.$$

First we solve

$$y'' + 3xy' + yy' - x^2 = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

We get a solution by Runge-Kutta applied to systems, say, $y(1) = 6$. Then we solve the same initial value problem but with $y'(0) = 5$. We get a new value of $y(1)$, say, $y(1) = 2$.

We compare the two values and try again, with $y'(0) = 4.5$ (When we tried $y'(0) = 0$, we got 6 and when we tried $y'(0) = 5$, we got 2. We used interpolation to figure that we should try 4.5. Suppose this time we got $y(1) = 3.2$. We then interpolate again to try a new value of $y'(0)$ and keep doing this until we are close enough to 3.

The finite difference method used the fact that, by Taylor's theorem,

$$y''(x) \approx \frac{y(x+h) + y(x-h) - 2y(x)}{h^2}$$

and that $y'(x) \approx (y(x+h) - y(x))/h$. We subdivide our interval $[a, b]$ into equal subintervals of length $1/n$, and call the partition points $x_i$. Let $y_i = y(x_i)$. To solve the above equation, we write $y_0 = 1$, $y_n = 3$, and

$$n^2(y_{i+1} + y_{i-1} - 2y_i) + 3nx_i(y_{i+1} - y_i) + ny_i(y_{i+1} - y_i) = x_i^2,$$

$i = 1, 2, \ldots, n-1$. We get a system of $n+1$ equations in $n+1$ variables. This might not be linear, if the differential equation is not linear, but there are methods for solving these. If our original differential equation is linear, the system will be linear and there are good methods for solving large systems of equations.

We now look at the Rayleigh-Ritz method. Let us start with an example.

$$-\frac{d}{dx}\Big((1+x^2)\frac{dy}{dx}\Big) + 3y = x^2, \qquad y(0) = 0, \quad y(1) = 0.$$

We could rewrite this as

$$-(1+x^2)y'' - 2xy' + 3y = x^2.$$

We'll show in a minute that the solution to this ODE minimizes the functional

$$I[u] = \int_0^1 [(1+x^2)u'(x)^2 + 3u(x)^2 - 2x^2 u(x)]\,dx.$$

(Because of the negative sign, the minimum is not necessarily $u = 0$.)

If we accept this for the moment, we minimize over a smaller class of functions $\sum_{j=1}^n c_j g_j$. For example, we can let the $g_j$ look like little tents: 0 if $|x - x_j| > 1/n$, $g(x_j) = 1$, and linear in between. Then

$$I[u] = \int_0^1 \Big[(1+x^2)\Big[\sum c_j g_j'(x)\Big]^2 + 3\Big[\sum c_j g_j(x)\Big]^2 - 2x^2\Big[\sum c_j g_j(x)\Big]\Big]\,dx.$$

We choose the $c_j$ to minimine this, so the partial of $I$ with respect to each $c_i$ is 0. Thus

$$0 = \frac{\partial I}{\partial c_i} = \int_0^1 \Big[(1+x^2)2\Big(\sum_j c_j g_j'(x)\Big)g_i'(x)$$
$$+ 3\cdot 2\Big(\sum_j c_j g_j(x)\Big)g_i(x) - 2x^2 g_i(x)\Big]\,dx = 0,$$

So

$$\sum_j c_j\Big[\int_0^1 \Big((1+x^2)2g_j'(x)g_i'(x) + 6g_j(x)g_i(x) - 2x^2 g_i(x)\Big)\,dx\Big] = 0$$

for each $j$. We know the $g_j$'s, so we can write this as

$$\sum_j A_{ij} c_j = 0,$$

which is a linear system of equations, and we can solve this for the $c_j$'s.

Why does $y$ minimize $I[u]$? Let $u$ be the minimum and look at $u + hv$, where $h \in \mathbb{R}$ and $v(0) = v(1) = 0$.

$$\begin{aligned} I[u+hv] = \int_0^1 \Big\{ &(1+x^2)(u')^2 + 2h(1+x^2)u'v' + (1+x^2)(v')^2 \\ &+ 3u^2 + 6huv + 3h^2v^2 \\ &- 2x^2u - 2hx^2v \Big\}\, dx. \end{aligned}$$

Since the minimum is at $u$, then as a function of $h$, $I[u+hv]$ takes its minimum at $h = 0$. So we take the derivative of $I[u+hv]$ with respect to $h$, set $h$ equal to 0, and we should get 0. Thus

$$\left.\frac{\partial I[u+hv]}{\partial h}\right|_{h=0} = \int_0^1 (2(1+x^2)u'v' + 6uv - 2x^2v)\, dx = 0.$$

Divide by 2 and then integrate by parts to get

$$\int_0^1 [-((1+x^2)u')' + 3u - x^2] v\, dx = 0.$$

We used the fact that $v(0) = v(1) = 0$ so as to not get any other terms. This holds for every $v$, and the only way that can happen is if

$$-((1+x^2)u')' + 3u - x^2 = 0.$$

In general, if we want to solve

$$-(p(x)y')' + q(x)y - r(x) = 0,$$

we minimize

$$I[u] = \int_a^b [p(x)(u'(x))^2 + q(x)(u(x))^2 - 2r(x)u(x)]\, dx.$$

## 15.3 Partial differential equations

The two methods people use are the finite differences method and the finite element method. For finite differences, one approximates

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{u(x+h,t)+u(x-h,t)-2u(x,t)}{h^2},$$

$$\frac{\partial u}{\partial t}(x,t) = \frac{u(x,t+k)-u(x,t)}{k},$$

etc. This method is good for equations like the heat equation.

The finite element is the same idea as the Rayleigh-Ritz method. One writes the solution as a function that minimizes a certain integral. This is good for boundary value problems.