FIBER BROWNIAN MOTION AND THE “HOT SPOTS” PROBLEM

Richard F. Bass and **Krzysztof Burdzy**

Abstract. We show that in some planar domains both extrema of the second Neumann eigenfunction lie strictly inside the domain. The main technical innovation is the use of “fiber Brownian motion,” a process which switches between two-dimensional and one-dimensional evolution.

1. Introduction. The main purpose of this article is to give a stronger counterexample to the “hot spots” conjecture than the one presented in Burdzy and Werner (1999). Along the way we will define and partly analyze a new process, which we call “fiber Brownian motion,” and which may have some interest of its own.

Consider a Euclidean domain which has a discrete spectrum for the Laplacian with Neumann boundary conditions, for example, a bounded domain with Lipschitz boundary. Recall that the first Neumann eigenfunction is constant. The “hot spots” conjecture says that the maximum of the second Neumann eigenfunction is attained at a boundary point. Burdzy and Werner (1999) constructed a domain where the second eigenfunction attains its maximum inside the domain but its minimum lies on the boundary. If φ is an eigenfunction, so is $-\varphi$ and hence the maximum and minimum are indistinguishable in the context of this problem. Hence, the counterexample of Burdzy and Werner (1999) leaves open the following question.

Question 1.1. *Must at least one of the extrema of the second Neumann eigenfunction be attained on the boundary of the domain?*

The uncertainty about the answer to this question is underscored by the nature of Burdzy and Werner’s counterexample, which cannot be easily modified to solve Question 1.1. The second author learned about a different counterexample, obtained by D. Jerison

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and N. Nadirashvili (private communication), which seems to have the same property that the minimum of the eigenfunction lies on the boundary. The answer to Question 1.1 is given in the following statement.

Theorem 1.2. *There exists a bounded Lipschitz domain in the plane such that its second Neumann eigenvalue is simple, and both extrema of the corresponding eigenfunction are attained only at interior points of the domain.*

We believe that each extremum of the eigenfunction in Theorem 1.2 is attained at a single point but we do not prove this.

Our example is based on an idea of Wendelin Werner involving two triangles whose vertices were connected by very thin tubes.

The "hot spots" conjecture was proposed by J. Rauch at a conference in 1974. The only published statement of the conjecture is contained in a book by Kawohl (1985). Bañuelos and Burdzy (1999) have proved the conjecture for a class of planar domains, and Burdzy and Werner (1999) found a counterexample. The introduction to the paper of Bañuelos and Burdzy (1999) contains a discussion of many aspects of the conjecture. Jerison and Nadirashvili (1999) have some new results on the "hot spots" problem in convex planar domains with two axes of symmetry.

The "fiber Brownian motion" is a process which moves like two-dimensional Brownian motion in a part of its state space, but it evolves like a one-dimensional Brownian motion if it happens to be on one of many "fibers" in its state space. The process is obtained as a weak limit of reflected Brownian motions in domains with very thin tubes. The main property of fiber Brownian motion used in the present article is that its motion along the fibers is not affected by their curvature, loosely speaking. This holds only for some families of fibers and leads to the natural question about the behavior in a general family of fibers. We will limit our analysis to the families of fibers needed to prove Theorem 1.2.

Many elements of the construction of the domain in Theorem 1.2 and also many elements of the argument are adapted from Burdzy and Kendall (2000), Bañuelos and Burdzy (1999) and Burdzy and Werner (1999). Two types of couplings, "synchronous" and "mirror," were used by Bañuelos and Burdzy (1999) to prove some results about "hot spots." We would like to emphasize that the simpler "synchronous" coupling does not work in the present paper and we have to use the more interesting "mirror" coupling, whose properties relevant to this problem were first analyzed by Burdzy and Kendall (2000).

The letter c with a subscript will denote a finite and strictly positive constant whose exact value is unimportant. We renumber the constants in each proof. In a number of places explicit constants appear, such as 223, 17, etc. These numbers were convenient for presenting the proofs, but are not otherwise important.

The next section contains an overview of our techniques and details about the organization of the paper.

We are grateful to M. van den Berg and W.D. Evans for very useful advice on spectral theory. We would also like to thank the referee for helpful suggestions on the presentation of our results.

2. Overview of proof.

Since our argument is rather technical, we will review in this section some known facts and techniques related to the "hot spots" problem and explain how they motivated our project. We will also outline the main components of our proofs. The precursor of the counterexample given in Burdzy and Werner (1999) was an idea of Wendelin Werner, which consists of two large triangles whose respective vertices are connected by very thin tubes. Consider the heat problem in this region with Neumann boundary conditions. One can think about heat and cold as two substances which avoid or anihilate each other. For this reason, if the initial temperature distribution is such that one of the triangles is hot and the other is cold, it seems natural that the first one will stay warmer than the other one forever. The center of the warmer triangle will be the hottest spot because the center is the point which lies at the greatest distance from the tubes. By symmetry, the center of the other triangle will be the coldest spot. An eigenfunction expansion translates this property of the heat equation solution into the fact that the second eigenfunction for the Neumann Laplacian attains its extrema at the centers of the triangles.

We will now explain how the above sketch of an argument may be implemented in a rigorous way when the triangles and tubes are on the surface of a two-dimensional manifold embedded in three-dimensional space. Consider a cylinder of finite height in $\mathbb{R}^3$ whose base is an equilateral triangle with each vertex truncated. The sides of the cylinder consist of six rectangles, three of which are very narrow and three of which are wider. The domain $\mathcal{D}_1$ will consist of part of the surface of this cylinder, namely, the top, bottom, and the three narrow rectangles on the sides; see the shaded area in Fig. 2.1.

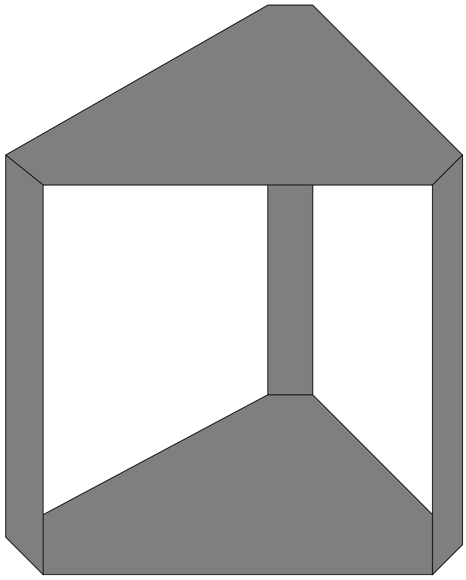

Figure 2.1.

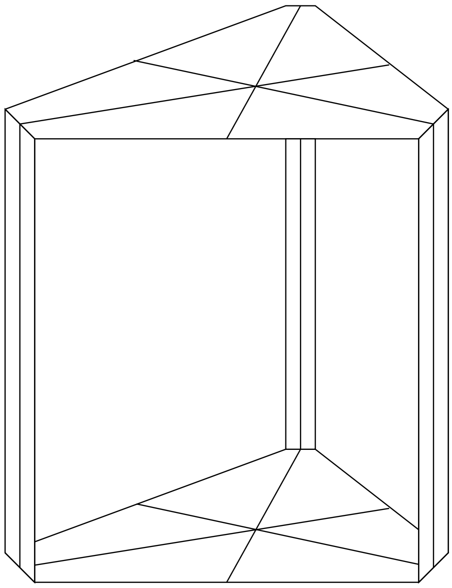

Figure 2.2.

The reason for distorting the Werner idea in this way is to generate more symmetries. We parenthetically note that the "hot spots" conjecture has not been studied in the context of manifolds as it is evidently false—the surface of an (American) football with a small hole far from its endpoints seems to be an obvious counterexample, although proving this assertion might not be quite trivial. We consider a manifold only to illustrate the method and some of the technical problems associated with it.

Split the surface $\mathcal{D}_1$ in Fig. 2.1 into 6 isometric parts as indicated in Fig. 2.2. Let one of them be called E; unfolded, E appears as in Fig. 2.3.

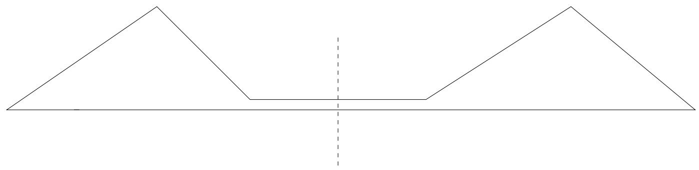

Figure 2.3.

It is not hard to show that if the tube is thin enough, then the second Neumann eigenfunction φ_E^2 for E is simple and symmetric (see proof of Lemma 6.7). Consider the heat problem in E with the intial temperature equal to 1 on the left and -1 on the right half of E. Using a coupling argument as in Bañuelos and Burdzy (1999), one can show that the left part will always have higher temperature than the right part of E. An eigenfunction expansion then can be used to prove that the maximum of φ_E^2 must take place at the left vertex of E (cf. Lemma 4.1).

We can combine the six copies of E to form $\mathcal{D}_1$ and by symmetry construct the second eigenfunction $\varphi_{\mathcal{D}_1}^2$ for $\mathcal{D}_1$ from copies of φ_E^2. The extrema of $\varphi_{\mathcal{D}_1}^2$ are the same as those of φ_E^2 and so they lie in the interior of $\mathcal{D}_1$. One can see that the second eigenvalue for $\mathcal{D}_1$ must be simple, and we thus have an example where neither extremum of the unique second eigenfunction takes place on the boundary.

When one tries to construct a similar counterexample in a Euclidean domain, one is confronted with the lack of convenient symmetries. Burdzy and Werner (1999) considered a domain invariant under rotation by the angle $2\pi/3$, consisting of two bulky parts connected by thin tubes (their example is too complicated to be faithfully represented by a small picture). The price they had to pay was that one of the extrema of the second eigenfunction had to lie on the exterior boundary of the set. This left the complete resolution of the "hot spot" conjecture unsettled, as one could make an argument that the original conjecture should have been interpreted as saying that "at least one of the extrema of the second eigenfunction must lie on the boundary." The effective indistinguishability between a second eignfunction φ and $-\varphi$ makes this interpretation possible.

Our paper aims at giving a complete negative answer to the "hot spots" conjecture by going back to the original idea of Werner. Let us consider two equilateral triangles which are the same size, disjoint, and both have their lower side on the x axis. We would like to connect respective vertices with thin tubes in such a way that all three tubes are isometric. This, of course, is not possible. We could connect the respective vertices by curves of the same length but as far as the eigenvalues and eigenfunctions are concerned, we would be then in the same situation as just having two disjoint triangles. Probabilistically, we can interpret this as saying that a Brownian motion cannot enter the curves with positive probability.

We will instead connect the two triangles by a collection of curves or fibers (in fact, an uncountable collection of them). The concept is illustrated in Fig. 2.4, with an enlargement of a neighborhood of a vertex illustrated in Fig. 2.5.

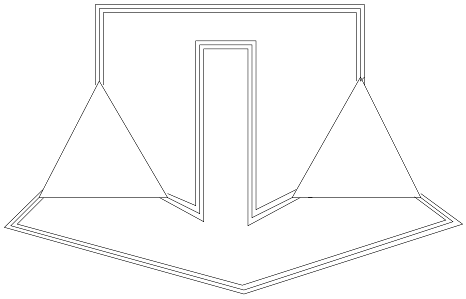

Figure 2.4.

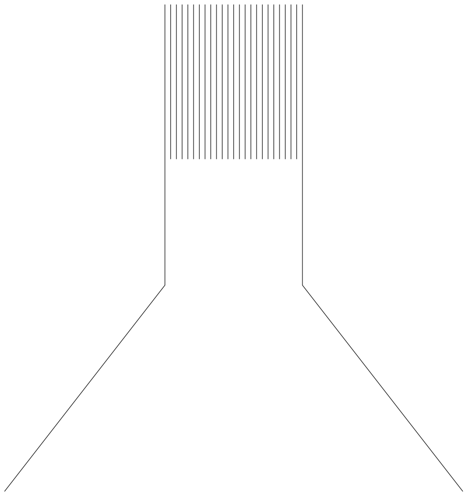

Figure 2.5.

A large number of obstacles are placed in the upper part of Fig. 2.5 so that a Brownian path can freely move in the lower part, but is forced to move in an almost vertical fashion between the obstacles. When the maximum distance between the obstacles goes to 0, the reflected Brownian motion in this domain will converge in distribution to a "fiber Brownian motion," a process which switches between 2-dimensional and 1-dimensional behavior.

If we call the set consisting of two triangles plus the connecting curves (fibers) $\mathcal{D}_2$, one can break $\mathcal{D}_2$ up into six isomorphic subsets, and proceed as in the discussion of $\mathcal{D}_1$ and E above. The difficulty is that we no longer have an open set. We therefore approximate $\mathcal{D}_2$ by a domain $\mathcal{D}_3$, which consists of the two triangles together with a large number of very thin tubes connecting neighborhoods of the respective vertices. All the tubes in $\mathcal{D}_3$ are approximately the same length. Much of the work in this paper is in constructing Brownian motion on $\mathcal{D}_2$ as the limit of Brownian motion in domains of the form of $\mathcal{D}_3$ (Sections 3 and 4) and showing that those spectral properties of $\mathcal{D}_3$ that are needed are close to the corresponding ones for $\mathcal{D}_2$ (Section 6).

We note that there exist at least two easy constructions of fiber-Brownian motion, as noticed by several of our colleagues. One of them uses a probabilistic technique of time-change and the other construction is based on excursion theory. We are forced to use a rather complicated approximation construction of fiber Brownian motion by the fact that in the end we have to consider a domain with very thin but not infinitely thin tubes.

Section 5 is a description of the coupling that is needed for our version of the Bañuelos and Burdzy results. We will use two types of couplings of Brownian paths, "synchronous" (see Fig. 2.6) and "mirror" (see Fig. 2.7).

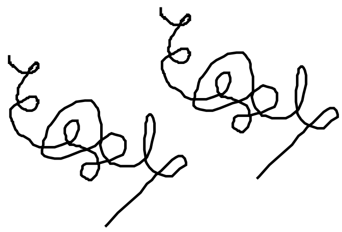

Figure 2.6.

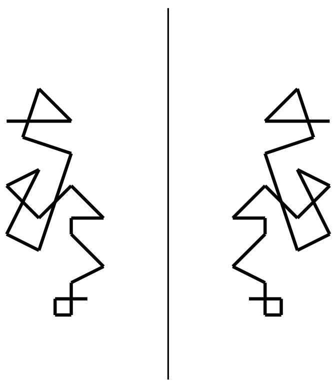

Figure 2.7.

These two basic coupling ideas are combined in a complicated way to obtain a new coupling which will always keep one of the particles in a slice (one sixth) of $\mathcal{D}_2$ to the left of the other particle, where "left" refers to a suitable partial order.

The order of presentation of our results is the following. We start with the approximation construction of fiber Brownian motion in Sections 3 and 4—these sections are independent of the rest of the paper and may have some interest of their own. Section 5 is devoted to the construction of a coupling and is also self-contained, but it has no relevance outside the scope of this paper. The bulk of Section 6 is devoted to various potential theoretic estimates which show, among other things, that we have uniform convergence for several crucial quantities when the domain $\mathcal{D}_2$ containing fiber bundle is approximated by domains $\mathcal{D}_3$ with thin tubes. The proof of our main result, Theorem 1.2, finishes the paper.

3. Brownian motion in thin tubes.

Remark 3.1. Several times we will need results about reflecting Brownian motion in a Lipschitz domain. A reference for these is Bass and Hsu (1991). That paper concerns domains in $\mathbb{R}^d$ with $d \geq 3$. However, given a domain D in $\mathbb{R}^2$ or a harmonic function $h(x_1, x_2)$ on D, one can easily derive the needed results from those in Bass and Hsu by considering $\widetilde{D} = D \times \mathbb{R}$, $\widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2)$, and $\widetilde{X}_t = (X_t, W_t)$, where W_t is a one dimensional Brownian motion independent of a two dimensional Brownian motion X_t.

Since the domains considered in this section have polygonal boundaries, we do not need the full strength of the results of Bass and Hsu (1991). We will mostly only sketch the proofs as elementary arguments are sufficient to support our claims.

For $\varepsilon \in (0, 1/2)$ and $\theta \in [0, \pi/3)$, define $V(\varepsilon, \theta)$ to be the domain bounded below by the curve $y = (\tan\theta)|x|$, above by the curve $y = (\tan\theta)|x| + \varepsilon/\cos\theta$, to the right by the line $y = -(\cot\theta)(x-1) + \tan\theta$, and to the left by the line $y = (\cot\theta)(x+1) + \tan\theta$. We will use $\mathcal{S}$ to denote the last two pieces of the boundary and we will refer to them as the sides of $V(\varepsilon, \theta)$. Let $X_t(\varepsilon, \theta)$ denote Brownian motion in $V(\varepsilon, \theta)$ with absorption on the sides of $V(\varepsilon, \theta)$ and normal reflection on the remainder of $\partial V(\varepsilon, \theta)$. When it is clear, we will write X instead of $X(\varepsilon, \theta)$ and V instead of $V(\varepsilon, \theta)$.

We begin by showing tightness for the family $\{X_t(\varepsilon, \theta)\}_{\varepsilon \in (0,1/2), \theta \in [0,\pi/3)}$.

Lemma 3.2. *Let $\varepsilon_n \to 0$ and let θ_n be a sequence in $[0, \pi/3]$. Let $\eta > 0$. There exists t independent of n such that*

$$\mathbb{P}^x(\sup_{s \leq t} |X_t(\varepsilon_n, \theta_n) - x| > \eta) < \eta, \qquad x \in V(\varepsilon_n, \theta_n), \quad n \geq 1.$$

Proof. Clearly we only need to consider n large enough so that $\varepsilon_n < \eta/8$. Let H_t^1 be the component of $X_t(\varepsilon_n, \theta_n)$ in the direction of the vector $(-1, \tan\theta)$ killed when the x component is greater than $-\eta/4$ and H_t^2 the component of $X_t(\varepsilon_n, \theta_n)$ in the direction $(1, \tan\theta)$ killed when the x component of $X_t(\varepsilon_n, \theta_n)$ is less than $\eta/4$. By a rotation, it is clear that both H_t^1 and H_t^2 are Brownian motions killed on hitting certain endpoints. No matter what x is, for $X_t(\varepsilon_n, \theta_n)$ to move more than η in time t, then either H_t^1 or H_t^2 must move at least $\eta/4$ in time t. We can make that probability as small as we like by taking t small enough. □

Let $B(x, r)$ denote the ball of radius r centered at x. Let

$$\sigma^{X(\varepsilon,\theta)}(x, r) = \inf\{t : X_t(\varepsilon, \theta) \in \partial B(x, r)\} \wedge \inf\{t : X_t \text{ hits } \mathcal{S}\};$$

we write $\sigma(x, r)$ when the process is understood.

Lemma 3.3. *There exists c_1 not depending on ε or θ such that $\mathbb{E}^x\sigma(x,r) \le c_1 r^2$ for all $x \in V(\varepsilon,\theta)$.*

Proof. Let X_t be represented as (X_t^1, X_t^2) in the usual coordinate system. Then the process $\widehat{X}_t = (|X_t^1|, X_t^2)$ is a reflected Brownian motion in the intersection $\widehat{V}(\varepsilon,\theta)$ of $V(\varepsilon,\theta)$ and the right half-plane, with killing on $\mathcal{S}$. Elementary estimates for Brownian motion, using Brownian scaling, show that for any starting point $x \in \widehat{V}(\varepsilon,\theta)$, the probability that $\widehat{X}_t$ does not hit $\partial B(x,r) \cup \mathcal{S}$ within the first r^2 units of time is less than some $p < 1$, independent of ε and θ. By the Markov property, the probability of not hitting $\partial B(x,r) \cup \mathcal{S}$ within the first kr^2 units of time is less than p^k. This easily implies the lemma.

We note that when $r \le 6\varepsilon$, the result follows from Bass and Hsu (1991) and scaling; see Remark 3.1. □

Lemma 3.4. *There exists c_1 such that if $\delta \in (0,1/2)$, then $\mathbb{E}^0 \int_0^\infty 1_{B(0,\delta)}(X_s)\,ds \le c_1\delta$, where c_1 is independent of ε and θ.*

Proof. From Lemma 3.3, starting in $B(0,\delta)$, the expected time before the process hits $\partial B(0,6\delta)$ is at most $c_2\delta^2$. By standard Brownian estimates, starting at $y \in \partial B(0,6\delta)$ there is probability at least $c_3\delta$ that the process will exit V through the sides $\mathcal{S}$ before returning to $B(0,\delta)$. By the results in Bass and Hsu (1991), $C_1 = \sup_{y\in B(0,\delta)} \mathbb{E}^y \int_0^\infty 1_{B(0,\delta)}(X_s)\,ds < \infty$. If $C_2 = \sup_{y\notin B(0,6\delta)} \mathbb{E}^y \int_0^\infty 1_{B(0,\delta)}(X_s)\,ds$, then we have

$$C_1 \le c_2\delta^2 + C_2.$$

Let τ be the lifetime of the process, i.e., the time when it hits $\mathcal{S}$. For any $y \notin B(0,6\delta)$,

$$\begin{aligned} C_2 &= \mathbb{E}^y\Big[\mathbb{E}^{X(\sigma(0,\delta))} \int_0^\infty 1_{B(0,\delta)}(X_s)\,ds; \sigma(0,\delta) < \tau\Big] \\ &\le C_1 \mathbb{P}^y(\sigma(0,\delta) < \tau) \\ &\le C_1(1 - c_3\delta). \end{aligned}$$

We obtain

$$C_1 \le c_2\delta^2 + C_1(1 - c_3\delta),$$

which yields $C_1 \le c_2\delta/c_3$. This completes the proof. □

Next we want to show the Hölder continuity of harmonic functions in V. Saying a function is harmonic in a subdomain of V means that it is harmonic in the interior of the subdomain and that it has normal derivative zero on the portion of ∂V at which X_t is reflecting.

Lemma 3.5. *Suppose for some z and r the function h is harmonic in $B(z,2r)\cap V(\varepsilon,\theta)$. Assume that $B(z,2r)\cap\mathcal{S}=\emptyset$. There exist c_1 and α not depending on r,ε or θ such that*

$$|h(x)-h(y)|\le c_1\Big(\sup_{B(z,2r)\cap V}|h|\Big)\Big(\frac{|x-y|}{r}\Big)^\alpha,\qquad x,y\in B(z,r)\cap V.$$

Proof. Let $\operatorname{Osc}_A h=\sup_A h-\inf_A h$. Suppose h is harmonic in $B(z,2r)\cap V$. We will show there exists $\rho<1$ independent of z,r,ε, and θ such that

$$\operatorname*{Osc}_{B(z,r)\cap V} h\le\rho\operatorname*{Osc}_{B(z,2r)\cap V} h, \tag{3.1}$$

and the result follows easily from this, with $\alpha=-\log\rho/\log 2$.

By looking at $Ah+B$ for suitable constants A and B, it suffices to prove (3.1) when $\sup_{B(z,2r)\cap V} h=1$ and $\inf_{B(z,2r)\cap V} h=0$. Note $V\cap\partial B(z,r)$ will consist of either one, two or three arcs. At least one of these arcs, say γ, will have length greater than $\varepsilon/2\wedge\pi r/2$. An arc of this size is quite a large target for Brownian motion and standard arguments can be used to show that

$$\mathbb{P}^y(X_t \text{ hits } \gamma \text{ before exiting } B(z,2r))\ge c_3,\qquad y\in B(z,r). \tag{3.2}$$

Pick a point $z_0\in\gamma$, and by looking at $1-h$ if necessary, we may suppose $h(z_0)\ge 1/2$. By the Harnack inequality for functions harmonic with respect to reflecting Brownian motion (see Bass and Hsu (1991), Theorem 3.9), $h\ge c_2$ on γ. Since h is harmonic, $h(y)\ge c_3(\inf_\gamma h)\ge c_2c_3$ for all $y\in B(z,r)$, and we then have

$$\operatorname*{Osc}_{B(z,r)\cap V} h=\sup_{B(z,r)\cap V} h-\inf_{B(z,r)\cap V} h\le 1-c_2c_3=(1-c_2c_3)\operatorname*{Osc}_{B(z,2r)\cap V} h,$$

which is (3.1) with $\rho=1-c_2c_3$. □

Recall that the Hausdorff distance between sets A and B is defined to be $\inf\{s: A\subset B^s, B\subset A^s\}$, where $B^s=\{x:\operatorname{dist}(x,B)<s\}$.

Theorem 3.6. *Suppose ε_n is a sequence tending to 0, $\theta_n\in[0,\pi/3]$. There exists a subsequence n_k with the following properties.*

(a) $V(\varepsilon_{n_k},\theta_{n_k})$ converges in the Hausdorff metric to a set V which is symmetric about the y axis and which in the first quadrant is a line segment starting from the origin.

(b) There exists a strong Markov process $(\mathbb{P}^z,X_t)$ on V such that whenever $x_n\in V(\varepsilon_n,\theta_n)\to x\in V$, then the $\mathbb{P}^{x_{n_k}}$ law of $X(\varepsilon_{n_k},\theta_{n_k})$ converges weakly to the $\mathbb{P}^x$ law of X_t.

(c) *Let* $\Phi : V \to \mathbb{R}$ *be defined by* $\Phi(x_1, x_2) = \mathrm{sgn}\,(x_1)\mathrm{dist}\,((x_1, x_2), (0, 0))$*. Then* $\Phi(X_t)$ *has the law of a one-dimensional Brownian motion killed on hitting* $\sec\theta$ *or* $-\sec\theta$*.*

Proof. If we start by taking a subsequence such that θ_{n_j} converges, say to θ_0, then (a) is routine.

To prove (b), we follow Barlow and Bass (1989), Section 6 (see Propositions 6.3–6.8). Here we give a sketch. Tightness has been shown in Lemma 3.2. With the estimates of Lemma 3.3 and 3.5, we then can prove equicontinuity of potentials as follows; cf. Barlow and Bass (1989), Section 5. Let us write V_n for $V(\varepsilon_n, \theta_n)$, X_t^n for $X_t(\varepsilon_n, \theta_n)$, and let τ_n be the first time X_t^n exits V_n through the sides $\mathcal{S}$. By Lemma 3.3, $c_1 = \sup_{x\in V_n} \mathbb{E}^x\tau_n < \infty$. If g is a bounded function, let $R_n g(x) = \mathbb{E}^x \int_0^{\tau_n} g(X_s^n)\,ds$. Note that $|R_n g(x)| \le c_1\|g\|_\infty$. If z is fixed, then for $x \in B(z, r)$,

$$R_n g(x) = \mathbb{E}^x \int_0^{\sigma(z,2r)} g(X_s^n)\,ds + \mathbb{E}^x R_n g(X^n(\sigma(z, 2r))).$$

A similar formula holds for $R_n g(y)$, for every $y \in B(z, r)$. The first term on the right is less than $c_2(2r)^2(\sup_{B(z,2r)} |g|)$ by Lemma 3.3. The second term is harmonic in $B(z, 2r)$, and so by Lemma 3.5

$$|\mathbb{E}^x R_n g(X^n(\sigma(z, 2r)) - \mathbb{E}^y R_n g(X^n(\sigma(z, 2r))| \le c_3|x - y|^\alpha r^{-\alpha}\|Rg\|_\infty, \qquad x, y \in B(z, r).$$

Therefore,

$$|R_n g(x) - R_n g(y)| \le c_4(r^2 + |x - y|^\alpha r^{-\alpha})\|g\|_\infty, \qquad x, y \in B(z, r).$$

Letting $r = c_5\sqrt{|x - y|}$, where c_5 is chosen so that $r > 2|x - y|$, we see that $R_n g$ is Hölder continuous with a modulus of continuity that depends only on the L^∞ bound on g. Using the resolvent equation

$$R_n^\lambda g = \sum_{i=0}^\infty \lambda^i R_n^{i+1} g,$$

where $R_n^\lambda g(x) = \mathbb{E}^x \int_0^{\tau_n} e^{-\lambda t} g(X_s^n)\,ds$, we see that as long as $\lambda < c_1^{-1}$, then $R_n^\lambda g$ is Hölder continuous with a modulus depending only on $\|g\|_\infty$.

Now take a countable dense subset $\{g_i\}$ of the continuous functions on $B(0, 100)$ and take a countable dense subset $\{\lambda_m\}$ of $(0, c_1^{-1})$. For every fixed i and m, in view of the uniform Hölder continuity of the functions in the sequence $\{R_n^{\lambda_m} g_i\}_n$, we can find a subsequence n_j such that the sequence $\{R_{n_j}^{\lambda_m} g_i\}_j$ converges. By a diagonalization process, passing to a subsequence if necessary, we obtain a single sequence n_j such that $R_{n_j}^{\lambda_m} g_i$ converges for each i and m. By approximating functions g with g_i and reals λ with λ_m, we can show that $R_{n_j}^\lambda g$ converges for each continuous g and each $\lambda \in (0, c_1^{-1})$. If we call the limit $R^\lambda g$, this is enough to show that there is a limit that is a strong Markov process; see Barlow and Bass (1989) for details.

The fact that the laws of X^n starting at x are tight for each x is sufficient to prove that the limit process is continuous; again see Barlow and Bass (1989) for details.

It remains to prove (c). For the process X_t when it is away from the origin, it is not hard to see that the limit is a Brownian motion; we thus have a diffusion on V, and under the map Φ we have a diffusion on $[-\sec\theta, \sec\theta]$. It is obvious that starting at the origin, the law of $\Phi(X_t)$ is the same as the law of $-\Phi(X_t)$, so $\Phi(X_t)$ is on natural scale. By Lemma 3.4, letting $\varepsilon \to 0$, we obtain for every $\delta \in (0, 1/2)$,

$$\mathbb{E}^0 \int_0^\infty 1_{[-\delta,\delta]}(\Phi(X_s))\, ds \le c_2 \delta.$$

Next we let $\delta \to 0$ to see that

$$\mathbb{E}^0 \int_0^\infty 1_{\{0\}}(\Phi(X_s))\, ds = 0. \tag{3.3}$$

This shows that $\Phi(X_t)$ does not spend positive time at the origin, hence the speed measure for $\Phi(X_t)$ is the same as that of Brownian motion. □

Remark 3.7. If we translate the family $V(\varepsilon, \theta)$ by an arbitrary vector and rotate it by an arbitrary angle, Theorem 3.6 will still hold for the new family, by the invariance of Brownian motion under such transformations.

We now construct a family of one-dimensional Brownian motions that connect a horizontal line segment with a vertical line segment by fibers of the same length. This is not hard to do, but we also want our Brownian motions to be the limit of reflecting Brownian motion in a planar domain.

For $v \in [0, 1]$ define $u(v)$ to be larger than v and to satisfy

$$21 + 4(1-v) + 2\sqrt{(u(v)-v)^2 + (99+v)^2} = 223.$$

Define the curve γ_v as follows. The curve begins at $P_0(v) = (v, 0)$, rises vertically to $P_1(v) = (v, 21-v)$, goes horizontally to $P_2(v) = (3-v, 21-v)$, goes down diagonally to $P_3(v) = (102, 21-u(v))$, goes up diagonally to $P_4(v) = (201+v, 21-v)$, and then goes horizontally to $P_5(v) = (203, 21-v)$. It is easy to see that each curve γ_v has length 223. and that the curves are piecewise linear and are pairwise disjoint.

For $n > 1$ define $L_1(v, n)$ to be the line that is parallel to the line segment that goes from P_2 to P_3, lies above that line segment, and is a distance 2^{-n} from that line segment. Similarly, let $L_2(v, n)$ be the line that is parallel to and a distance 2^{-n} above the line segment that goes from P_3 to P_4. Let $L_3(v, n)$ be the line $\{y = 21 - v + 2^{-n}\}$. Let

$R_2(v,n)$ be the point of intersection of $L_1(v,n)$ and the line $L_3(v,n)$, let $R_3(v,n)$ be the point of intersection of $L_1(v,n)$ and $L_2(v,n)$, and let $R_4(v,n)$ be the point of intersection of $L_2(v,n)$ and the line $L_3(v,n)$. Let ψ_v be the curve that starts at $R_0(v,n) = (v-2^{-n}, 0)$, rises vertically to $R_1(v,n) = (v - 2^{-n}, 21 - v + 2^{-n})$, goes horizontally to $R_2(v,n)$, goes diagonally to $R_3(v,n)$, goes diagonally to $R_4(v,n)$, and then goes horizontally to $R_5(v,n) = (203, 21 - v + 2^{-n})$. Define $I(v,n)$ to be the domain bounded by γ_v, ψ_v, the line $y = 0$ and the line $x = 203$.

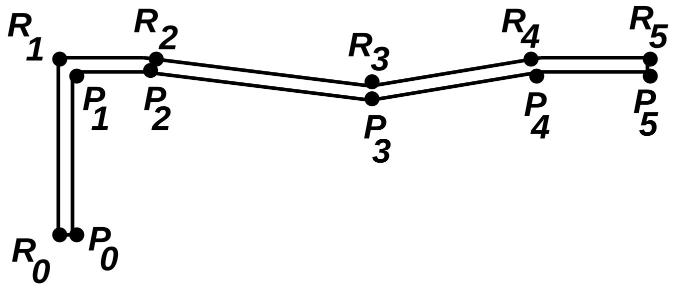

Figure 3.1.
The set $I(v,n)$. The figure is not to scale.

Let

$$D_n = \bigcup_{i=1}^{2^n} I(i/2^n, n).$$

Some elementary geometry shows that $I(i/2^n, n)$ and $I(j/2^n, n)$ are disjoint if $i \neq j$.

Define $X_t(n)$ to be Brownian motion in D_n with absorption on the portion of the boundary that is a subset of $y = 0$ and $x = 203$, and with normal reflection everywhere else that the normal derivative is defined.

Set $D = \cup_{x \in [0,1]} \gamma_x$.

Theorem 3.8. *There exists a subsequence n_k with the following properties.*

(a) *There exists a strong Markov process $(\mathbb{P}^z, X_t)$ on D such that whenever $x_{n_k} \to x \in D$, then the $\mathbb{P}^{x_{n_k}}$ law of $X(n_k)$ converges weakly to the $\mathbb{P}^x$ law of X.*

(b) *Starting at $z \in \gamma_x$, the $\mathbb{P}^x$-law of X_t is that of a Brownian motion on the curve γ_x.*

Proof. There were four ingredients to the proof of Theorem 3.6: (i) a tightness estimate; (ii) an estimate on $\mathbb{E}^y \sigma(x,r)$; (iii) the Hölder continuity of harmonic functions; and (iv) the identification of the limit.

Note that each of these four ingredients is a local property. Since our domain D_n is the union of $I(i/2^n, n)$, none of which is accessible from any of the other $I(j/2^n, n)$ by a Brownian motion, and each $I(i, 2^n)$ is the union of sets that are rotations of the $V(\varepsilon, \theta)$'s considered above, we apply the methods of Theorem 3.6 together with Remark 3.7 to obtain our proof. □

Remark 3.9. We have connected a horizontal line segment with a vertical one. The same technique with only minor modifications allows us to connect a horizontal line segment with one of any angle whatsoever.

4. Fiber Brownian motion in a rectangle.

Let $D = [-1,1] \times [0,1]$. In this section we want to construct a process that behaves like planar Brownian motion in the right half of D and moves only horizontally in the left half of D. Martin Barlow and Ron Pyke pointed out to us that one way of doing this is to take two-dimensional Brownian motion in D and time change it by the inverse of the additive functional that increases according to the amount of time spent in the right half of D. We take the limit approach here because of the application we have in mind.

Define D_n to be $[-1,1] \times [0,1]$ with the line segments $[-1,0] \times \{i/2^n\}$ removed for $i = 0, 1, \ldots, 2^n$. Let $X_t(n)$ be reflecting Brownian motion in D_n with absorption on the lines $x = -1$ and $x = 1$.

First we show tightness for the family of processes $\{X_t(n)\}_{n \geq 1}$.

Lemma 4.1. *Given $\eta > 0$ there exists t such that $\mathbb{P}(\sup_{s \leq t} |X_s(n) - x| > \eta) < \eta)$ for all $x \in D_n$ and all n.*

Proof. Since the horizontal component of $X_t(n)$ is clearly a one-dimensional Brownian motion, we can focus on the vertical component. Let $L_i = [0,1] \times \{i/2^n\}$. Let $T_1 = \inf\{t : X_t(n) \in L_i \text{ for some } i\}$, and define T_{i+1} to be the first time $X_t(n)$ is in some L_j other than the one $X_{T_i}(n)$ is in. By symmetry, $X_{T_i}(n)$ is a symmetric random walk. Let N be the number of steps it takes this random walk to move vertically either plus or minus η.

Note $T_{i+1} - T_i$ is stochastically larger than the time it takes a standard one-dimensional Brownian motion to move $1/2^n$. Also, the T_i are independent of N. So the time for $X_t(n)$ to move vertically $\pm\eta$ is stochastically larger than the time for a standard Brownian motion to move $\pm\eta$. Tightness is an immediate consequence. □

The estimate $\mathbb{E}^y \sigma(x,r) \leq c_1 r^2$ is easy because the horizontal component is a one-dimensional Brownian motion and we have the estimate for such a process.

Let $Q((x_1,x_2),r) = [x_1 - r, x_1 + r] \times [x_2 - r, x_2 + r]$. For $x, y \in D$, define $d_n(x,y) = \inf\{\text{length}(\gamma)\}$, where the infimum is over all curves γ connecting x and y and contained in the interior of D_n (except, possibly, for x and y).

Lemma 4.2. *Suppose h is harmonic in $Q(z,2r)\cap D_n$. There exist c_1 and α not depending on z,r, or n such that for every connected component $\widehat{Q}$ of $Q(z,r)\cap D_n$,*

$$|h(x)-h(y)| \le c_1\Big(\frac{d_n(x,y)}{r}\Big)^{\alpha}\big(\sup_{Q(z,2r)\cap D_n}|h|\big), \qquad x,y\in\widehat{Q}.$$

Proof. The proof will use coupling. We review the basic ideas and techniques of couplings at the beginning of Section 4, where they play the crucial role. The applications of couplings to estimates of harmonic functions in the current proof are quite standard so we will only outline the main steps of the argument.

Fix some connected component $\widehat{Q}$ of $Q(z,r)\cap D_n$ and suppose $x,y\in\widehat{Q}$. We start two reflected Brownian motions W^x and W^y in D_n, with W^x started at x and W^y started at y. We will define them on a common probability space so that they meet at a certain time T ("coupling time"). We will argue that the coupling time is small with large probability, for a suitably chosen pair W^x and W^y.

If $Q(z,3r/2)$ is disjoint from the vertical axis, we can rather easily couple W^x and W^y before they leave $Q(z,3r/2)$. This is because when $Q(z,3r/2)$ lies to the right of the y-axis, the processes W^x and W^y behave like standard Brownian motions until they leave the set. If $Q(z,3r/2)$ lies to the left of the y-axis, a mirror coupling of reflected Brownian motions in a strip gives us the desired result. In both cases, the processes can be coupled within $Q(z,3r/2)$ with probability greater than $p_1>0$, independent of n,z or r.

Now consider the case when $Q(z,3r/2)$ intersects the vertical axis. Let B_1 and B_2 be the balls with radii $r/8$ and $r/16$ and common center located $7r/4$ units to the right of z. With probability $p_2>0$, each of the processes W^x and W^y will hit B_2 within the first r^2 units of time, before leaving $Q(z,2r)$, and then stay inside B_1 for the next $2r^2$ units of time. Hence, assuming independent evolution of the processes at the initial stage, we see that with probability p_2^2 they will be in B_1 at the same time, say S, before exiting $Q(z,2r)$. After time S we coordinate their motions so that they meet at some time T, before leaving $Q(z,2r)$, with probability $p_3>0$. In all cases, we can couple the processes with probability $p_4>0$, at some time T less than the exit time $\tau_{Q(z,2r)}$ from $Q(z,2r)$.

Now suppose h is harmonic in $Q(z,2r)$. Let $\mathbb{P}^{(x,y)}$ be the joint law of (W^x,W^y). By looking at $Ah+B$ for suitable A and B, we may suppose $\sup_{Q(z,2r)} h = 1$ and $\inf_{Q(z,2r)} h = 0$. Let $\tau_{Q(z,2r)}$ be the first time the process exits $Q(z,2r)$. We have

$$\begin{aligned} h(x) &= \mathbb{E}^{(x,y)} h(W^x_{T\wedge\tau_{Q(z,2r)}}) \\ &= \mathbb{E}^{(x,y)}[h(W^x_T); T<\tau_{Q(z,2r)}] + \mathbb{E}^{(x,y)}[h(W^x_{\tau_{Q(z,2r)}}); \tau_{Q(z,2r)}<T]. \end{aligned}$$

We have a similar expression for $h(y)$. Note that on $(T<\tau_{Q(z,2r)})$ we have $W^x_T=W^y_T$. So, taking the difference and using the fact that $h\ge 0$,

$$h(x)-h(y) \le \big(\sup_{Q(z,2r)} h\big)\,\mathbb{P}^{(x,y)}(T>\tau_{Q(z,2r)}) \le 1-c_2.$$

Similarly $h(y) - h(x) \le 1 - c_2$. Therefore the oscillation of h in $Q(z,r)$ is less than $1 - c_2$. Just as in Lemma 3.5, this oscillation estimate easily implies the result. □

Let $\tau_n^* = \inf\{t : X_t(n) \in \{-1,1\} \times [0,1]\}$ and $R_n^\lambda g(x) = \mathbb{E}^x \int_0^{\tau_n^*} g(X_s(n))ds$. We will often write $R_n g$ for $R_n^0 g$.

Lemma 4.3. *There exist c_1 and α not depending on n such that*

$$|R_n g(x) - R_n g(y)| \le c_1 d_n(x,y)^\alpha \|g\|_\infty, \qquad x, y \in D_n.$$

Proof. This follows from Lemma 4.2 and the estimate $\mathbb{E}^y \sigma(x,r) \le c_2 r^2$, just as in the proof of Theorem 3.6. □

For fixed $x, y \in D$, the function $n \to d_n(x,y)$ is non-decreasing. Let $d_\infty(x,y) = \lim_{n\to\infty} d_n(x,y)$. Since $d_n(x,y) \le d_\infty(x,y)$ for all x, y and n, Lemmas 4.1-4.3 hold if we replace the Euclidean metric and d_n-metric with d_∞-metric in their statements.

In order to use the method of Barlow and Bass (1989) to construct a strong Markov process on D, it is necessary to use the Ascoli-Arzelá theorem. However D with the metric d_∞ is not separable. We thus need the following lemma.

Lemma 4.4. *Suppose g is continuous on $[-1,1] \times [0,1]$ with respect to the Euclidean metric. Then $R_n g$ is continuous on D_n with respect to the same metric. The modulus of continuity of $R_n g$ depends on g but it does not depend on n.*

Proof. Consider arbitrary x and v in D_n. Let y be a point with the same first coordinate as x and the same second coordinate as v. Lemma 4.3 provides an effective estimate for $|R_n g(v) - R_n g(y)|$ because $d_n(v,y)$ is the same as the Euclidean distance between v and y. Our goal is to find an estimate for $|R_n g(v) - R_n g(x)|$. Using the triangle inequality and an estimate for $|R_n g(v) - R_n g(y)|$ reduces the problem to estimating $|R_n g(x) - R_n g(y)|$.

If the first coordinate of x is positive, Lemma 4.3 provides again a desired estimate. It remains to consider the case when $x, y \in D_n \cap [-1,0) \times [0,1]$ and y lies directly above x.

We will use a coupling, as in the proof of Lemma 4.2. Let W^x and W^y be Brownian motions in D_n started at x and y, resp., with absorption on the left and right sides of D_n and reflection on the remainder of the boundary of D_n. We construct W^x and W^y so that their horizontal components are equal and their vertical components are independent. Let $T = \inf\{t : W_t^x \in \{-1,0\} \times [0,1]\}$. We have

$$R_n g(x) = \mathbb{E}^{(x,y)} \int_0^T g(W_s^x)ds + \mathbb{E}^{(x,y)} R_n g(W_T^x), \tag{4.1}$$

and similarly for y. Note that $R_n g(w) = 0$ for $w \in \{-1\} \times [0,1]\}$. Hence, if $W_T^x \in \{-1\} \times [0,1]\}$ then $R_n g(W_T^x) = R_n g(W_T^y) = 0$ and so $R_n g(W_T^x) - R_n g(W_T^y) = 0$. In the case when $W_T^x \in \{0\} \times [0,1]\}$, we will estimate the quantity $R_n g(W_T^x) - R_n g(W_T^y)$ using Lemma 4.3. Note that for $t \le T$, $|W_t^x - W_t^y| \le |x-y| + 2 \cdot 2^{-n}$ since the two horizontal components are equal and the width of the tubes is 2^{-n}. We see that

$$\begin{aligned}|\mathbb{E}^{(x,y)} R_n g(W_T^x) - \mathbb{E}^{(x,y)} R_n g(W_T^y)| &\le \sup_{|w-z| \le |x-y| + 2^{-n+1}} |R_n g(w) - R_n g(z)| \\ &\le c_2(|x-y| + 2^{-n+1})^\alpha \|g\|_\infty.\end{aligned} \tag{4.2}$$

It is easy to see that for some $c_3 < \infty$ independent of n, we have $\mathbb{E}^{(x,y)} T < c_3$. Then

$$\begin{aligned}&\Big|\mathbb{E}^{(x,y)} \int_0^T g(W_s^x)\, ds - \mathbb{E}^{(x,y)} \int_0^T g(W_s^y)\, ds\Big| \\ &\qquad \le \Big(\sup_{|w-z| \le |x-y| + 2^{-n+1}} |g(w) - g(z)| \Big) \mathbb{E}^{(x,y)} T \\ &\qquad \le c_3 \Big(\sup_{|w-z| \le |x-y| + 2^{-n+1}} |g(w) - g(z)| \Big).\end{aligned} \tag{4.3}$$

Using (4.1)-(4.3), we obtain

$$|R_n g(x) - R_n g(y)| \le c_2(|x-y| + 2^{-n+1})^\alpha \|g\|_\infty + c_3 \Big(\sup_{|w-z| \le |x-y| + 2^{-n+1}} |g(w) - g(z)| \Big).$$

This proves the lemma. □

In the next theorem the topology on D is the Euclidean one.

Theorem 4.5. *There exists a subsequence n_k with the following properties.*

(a) There exists a strong Markov process $(\mathbb{P}^z, X_t)$ on D such that whenever $x_n \to x \in D$, then the $\mathbb{P}^{x_{n_k}}$ law of $X_t(n_k)$ converges weakly to the $\mathbb{P}^x$ law of X_t.

(b) For every $x \in D$, the horizontal component of the process with distribution $\mathbb{P}^x$ is a Brownian motion. When the horizontal component is positive, the process behaves like a planar Brownian motion; when the horizontal component is negative, the vertical component does not change. The process is killed when its horizontal component reaches 1 or -1.

Proof. Since the horizontal component of every $\mathbb{P}^{x_n}$-process is a Brownian motion for each n, this is true in the limit as well. It is therefore clear that

$$M = \sup_n \sup_{x \in D_n} \mathbb{E}^x \tau_n^* < \infty.$$

By the resolvent equation, if $\lambda < 1/M$,

$$R_n^\lambda g = R_n^0 g - \lambda R_n^0 R_n^0 g + \lambda^2 R_n^0 R_n^0 R_n^0 g - \cdots.$$

By Lemma 4.4, each term in the sum is continuous with a modulus of continuity depending on g but not on n. Moreover, the i-th summand is bounded in absolute value by $\lambda^i M^{i+1} \|g\|_\infty$, so the series converges uniformly, independently of n. Hence for $\lambda < 1/M$, $R_n^\lambda g$ is continuous with a modulus depending on g but not on n.

With this observation and Lemma 4.1, the other parts of the theorem are proved as in the proof of Theorem 3.6. Note that the $\mathbb{P}^x$-process spends zero time on the vertical axis because its horizontal component is a Brownian motion. $\square$

5. Coupling of fiber Brownian motions. We start this section with a review of those properties of the "mirror coupling" for reflecting Brownian motions which are relevant to our argument. These aspects of the idea were originally developed by Burdzy and Kendall (2000) and later applied by Bañuelos and Burdzy (1999).

We start with the coupling of two Brownian motions in $\mathbb{R}^2$. Suppose that $x, y \in \mathbb{R}^2$ are symmetric with respect to a line M. Let X_t be a Brownian motion starting from x and let τ be the hitting time of M by X. We let Y_t be the mirror image of X_t with respect to M for $t \le \tau$, and we let $Y_t = X_t$ for $t > \tau$. The process Y_t is a Brownian motion starting from y. The pair (X_t, Y_t) is a "mirror coupling" of Brownian motions.

Next we turn to the mirror coupling of reflected Brownian motions in a half-plane $\mathcal{H}$, starting from $x, y \in \mathcal{H}$. Let M be the line of symmetry for x and y. The case when M is parallel to $\partial\mathcal{H}$ is essentially a one-dimensional problem, so we focus on the case when M intersects $\partial\mathcal{H}$. By performing rotation and translation, if necessary, we may suppose that $\mathcal{H}$ is the upper half-plane and M passes through the origin. We will write $x = (r^x, \theta^x)$ and $y = (r^y, \theta^y)$ in polar coordinates. The points x and y are at the same distance from the origin so $r^x = r^y$. Suppose without loss of generality that $\theta^x < \theta^y$. We first generate a 2-dimensional Bessel process R_t starting from r^x. Then we generate two coupled one-dimensional processes on the "half-circle" as follows. Let $\widetilde{\Theta}_t^x$ be a 1-dimensional Brownian motion starting from θ^x. Let $\widetilde{\Theta}_t^y = -\widetilde{\Theta}_t^x + \theta^x + \theta^y$. Let Θ_t^x be reflected Brownian motion on $[0, \pi]$, constructed from $\widetilde{\Theta}_t^x$ by the means of the Skorokhod equation. Thus Θ_t^x solves the stochastic differential equation $d\Theta_t^x = d\widetilde{\Theta}_t^x + dL_t$, where L_t is a continuous nondecreasing process that increases only when Θ_t^x is equal to 0 or π and Θ_t^x is always in the interval $[0, \pi]$. Thus Θ_t^x is constructed in such a way that the difference $\Theta_t^x - \widetilde{\Theta}_t^x$ is constant on every interval of time on which Θ_t^x does not hit 0 or π. The analogous reflected process obtained from $\widetilde{\Theta}_t^y$ will be denoted $\widehat{\Theta}_t^y$. Let τ^Θ be the smallest t with $\Theta_t^x = \widehat{\Theta}_t^y$. Then we let $\Theta_t^y = \widehat{\Theta}_t^y$ for $t \le \tau^\Theta$ and $\Theta_t^y = \Theta_t^x$ for $t > \tau^\Theta$. We define a "clock" by $\sigma(t) = \int_0^t R_s^{-2} ds$. Then $X_t = (R_t, \Theta_{\sigma(t)}^x)$ and $Y_t = (R_t, \Theta_{\sigma(t)}^y)$ are reflected Brownian motions in $\mathcal{H}$ with normal reflection—one can prove this using the same ideas as in the discussion of the skew-product decomposition for 2-dimensional Brownian motion presented by Itô and McKean (1974). Moreover, X_t and Y_t behave like free Brownian motions coupled by the mirror coupling as long as they are both strictly inside $\mathcal{H}$. The processes will stay together after the first time they meet. We call (X_t, Y_t) a "mirror coupling" of reflecting Brownian motions.

The two processes X_t and Y_t in the upper half-plane remain at the same distance from the origin. Suppose now that $\mathcal{H}$ is an arbitrary half-plane, and x and y belong to $\mathcal{H}$. Let M be the line of symmetry for x and y. Then an analogous construction yields a pair of reflecting Brownian motions starting from x and y such that the distance from X_t to $M \cap \partial\mathcal{H}$ is always the same as for Y_t. Let M_t be the line of symmetry for X_t and Y_t. Note that M_t may move, but only in a continuous way, while the point $M_t \cap \partial\mathcal{H}$ will never move. We will call M_t the *mirror* and the point $H = M_t \cap \partial\mathcal{H}$ will be called the *hinge*. The absolute value of the angle between the mirror and the normal vector to $\partial\mathcal{H}$ at H can only decrease.

The next level of generality is to consider a mirror coupling of reflected Brownian motions in a polygonal domain $\mathcal{D}$. Suppose that X_t and Y_t start from x and y inside $\mathcal{D}$ and let I be the side of $\mathcal{D}$ which is hit first by one of the particles. Let K be the straight line containing I. Since the process which hits I does not "feel" the shape of $\mathcal{D}$ except for the direction of I, it follows that the two processes will remain at the same distance from the hinge $H_t = M_t \cap K$. The mirror M_t can move but the hinge H_t will remain constant as long as I remains the side of $\partial\mathcal{D}$ where the reflection takes place. The hinge H_t will jump when the reflection location moves from I to another side of $\mathcal{D}$. The hinge H_t may from time to time lie outside $\partial\mathcal{D}$, if $\mathcal{D}$ is not convex.

We continue by defining a set D which carries a fiber Brownian motion and is amenable to coupling techniques. The first ingredient is a group $\mathcal{G}$ containing 6 elements, generated by (a) reflection with respect to the horizontal axis and (b) rotation around $(0,0)$ by the angle $2\pi/3$. We define a point-to-set mapping $\mathcal{T}x = \{\sigma(x),\ \sigma \in \mathcal{G}\}$. Typically, $\mathcal{T}x$ contains 6 points. The meaning of $\mathcal{T}K$ for a set K is self-evident.

Fix some $\mathbf{w} \in (0, 1/100)$ whose value will be chosen later. Consider points

$$A_0 = (0,0), \quad A_1 = (1, \sqrt{3}), \quad A_2 = (5, \mathbf{w}), \quad A_3 = (20, \mathbf{w}), \quad A_4 = (20, 0).$$

Let D_L^1 be the closed bounded set whose boundary is a polygonal line with consecutive vertices A_0, A_1, A_2, A_3, A_4, and A_0. Figure 5.1 presents a schematic drawing of the set D_L^1.

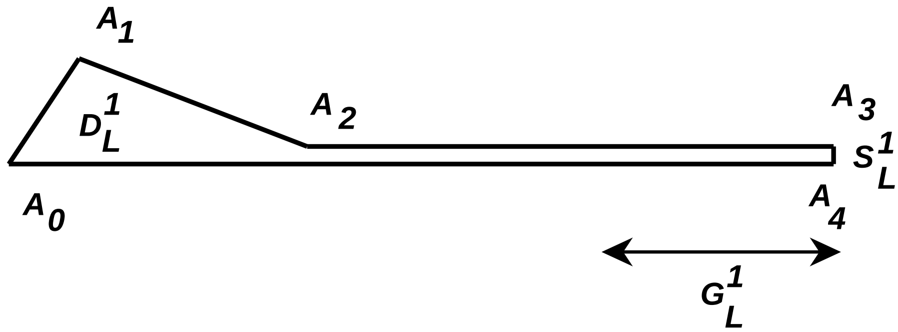

Figure 5.1.
The figure is not to scale.

Let $D_L = \mathcal{T}D_L^1$ and let D_R be the set that is symmetric to D_L with respect to the line $\mathcal{L} = \{(x,y) \in \mathbb{R}^2 : x = 30\}$. By abuse of notation $\mathcal{L}x$ and $\mathcal{L}Q$ will denote the point and set symmetric to x and Q, resp., with respect to $\mathcal{L}$.

The set D that we are trying to construct is the union of D_L, D_R, and three polygonal "tubes" T_1, T_2 and T_3 which will be described in a somewhat informal way. Let S_L^1 be the line segment between A_3 and $(20, -\mathbf{w})$. This line segment lies on the boundary of D_L. Let S_L^2 and S_L^3 be the images of S_L^1 by the rotations by angles $2\pi/3$ and $4\pi/3$. Let $S_R^k = \mathcal{L}S_L^k$, $k = 1,2,3$. The tubes T_k, $k = 1,2,3$, will be disjoint closed polygonal sets, symmetric with respect to $\mathcal{L}$, whose interiors are also disjoint from D_L and D_R. The tube T_k shares the piece of the boundary S_L^k with D_L and S_R^k with D_R.

Let $G_L^1 = \{(x,y) \in D_L : x \geq 14\}$ and let G_L^2 and G_L^3 be obtained by rotating G_L^1 by angles $2\pi/3$ and $4\pi/3$. Let $G_R^k = \mathcal{L}G_L^k$ and $G^k = G_L^k \cup G_R^k \cup T_k$, for $k = 1,2,3$, and $F = \bigcup_{k=1,2,3} G^k$. The results of the previous sections (see especially the construction of D_n preceding Theorem 3.8) show that T_k's can be chosen so that the following is true. For every $k = 1,2,3$, there exists a family of disjoint polygonal curves $\{\gamma_x^k\}_{x \in S_L^k}$ with the following properties.

(i) $\gamma_x^k \cap S_L^1 = \{x\}$,
(ii) for every x and k, the curve γ_x^k is $\mathcal{L}$-symmetric,
(iii) $\bigcup_{x \in S_L^k} \gamma_x^k = G^k$,
(iv) the set $\gamma_x^k \cap D_L$ is a line segment perpendicular to S_L^k,
(v) all curves γ_x^k, for all x and k, have the same length—we will denote it by α.

Moreover, we can choose the γ_x^k's so that the fiber Brownian motion defined in $D = D_L \cup D_R \cup \bigcup_{k=1,2,3} T_k$ is the limit of reflecting Brownian motions in a sequence of open approximating domains. We will be more precise about this point later on.

It may be necessary for the γ_x^k's, especially for the γ_x^1's, to make several turns in order to satisfy condition (v). This is fine as long as we have the approximation property mentioned in the last paragraph.

Note that although we can choose the γ_x^2's and γ_x^3's so that they are symmetric with respect to the horizontal axis, no such symmetry can exist between the γ_x^2's and γ_x^1's. This fundamental lack of symmetry in the picture is the main reason for using fiber Brownian motion instead of the usual reflected Brownian motion in an open domain.

Let $\widetilde{D} = D_L^1 \cup D_R^1 \cup \bigcup_{x \in D_L^1 \cap S_L^1} \gamma_x^1$.

We define a quantity representing the position of a point inside one of the long tubes in D. Every point $y \in F \cap D_R$ belongs to a unique fiber γ_x^k. Let $\rho(y)$ denote its distance from $\mathcal{L}$ along γ_x^k. For $x \in D_R \setminus F$ we let $\rho(x)$ be the sum of $\alpha/2$ and the Euclidean distance from x to $\bigcup_{k=1,2,3} S_R^k$. So far, we have defined $\rho(x)$ for $x \in D_R$. For $x \in D_L$ we let $\rho(x) = -\rho(\mathcal{L}x)$.

For $a > 0$ we let $F_a = \{x \in D : -a - \alpha/2 < \rho(x) < a + \alpha/2\}$, $F_a^c = D \setminus F_a$ and $\widetilde{F}_a = \widetilde{D} \cap F_a$.

We will consider a fiber Brownian motion in $\widetilde{D}$, i.e., a process which is a reflected Brownian motion in $\widetilde{D} \setminus F$ and it moves like a one-dimensional Brownian motion along the curves γ_x^1. A construction of such a process can be achieved combining the ideas presented in Sections 2 and 4. Section 2 explains how to define approximating domains with thin tubes which can bend and such that in the limit we obtain a family of polygonal lines and a one-dimensional Brownian motion along them. Section 3 shows how to define the transition between two-dimensional and one-dimensional behavior for the Brownian motion. Since the transitions are local in nature, the results obtained in that section for a rectangle can be generalized in an obvious way to $\widetilde{D}$.

Next we are going to introduce four Markov transition mechanisms for a pair of Brownian particles X_t and Y_t in $\widetilde{D}$. Later we will build a process which from time to time changes its transition mechanism and we will speak about the four "modes" of particle motion.

(A) If X_t and Y_t are in this mode, they behave like a pair of independent fiber Brownian motions.

(B) This mode is defined only if both particles stay in $\widetilde{D} \setminus F$. They behave like reflected Brownian motions in this set, related by mirror coupling. The processes are stopped when one of the particles hits the boundary of F.

(C) The third mode of motion is defined as long as both particles stay in $\widetilde{F}_{15}$. The particles move in such a way that $\rho(X_t) - \rho(Y_t)$ remains constant. They are stopped when one of them exits from $\widetilde{F}_{15}$. There are many two-particle processes with this property. For definiteness, we assume that when a particle is in $D_L \cap (\widetilde{F}_{15} \setminus F)$ then its component perpendicular to the long side of this set is independent of the motion of the other particle, and the same is true when a particle is in $D_R \cap (\widetilde{F}_{15} \setminus F)$.

(D) In this mode we have $X_t = Y_t$.

We leave a formal proof of existence of processes satisfying (A)-(D) to the reader. To each of the modes of motion (A)-(D) we associate a stopping time.

T^A: The first time t such that $|\rho(X_t) - \rho(Y_t)| \leq 6$.
T^B: The first time t such that $|\rho(X_t) - \rho(Y_t)| \geq 7$ or $X_t = Y_t$.
T^C: The first time t such that one of the processes X_t or Y_t hits $\widetilde{D} \setminus \widetilde{F}_{15}$.
T^D: Infinity.

Our coupled processes will move according to the following rules. We will assume that the starting location $(X_0, Y_0) = (x_0, y_0)$ is confined to a subset Q of $\widetilde{D} \times \widetilde{D}$, to be described below. The processes start moving in one of the modes (A)-(C), depending on (x_0, y_0). If the mode is (A), the particles move until the time T^A and then they switch to either (B) or (C), according to the following rule. If both particles are in $\widetilde{F}_{15}$, then they switch to mode (C). Otherwise they switch to (B). Whenever the particles are in mode (B), they wait until T^B and then they switch to (A) if $|\rho(X_t) - \rho(Y_t)|$ attains the value 7. Otherwise, if the particles meet at some time, the processes switch to mode (D). If they are in mode (C), then they wait until T^C and switch at this time to (B). Finally, they never make a transition out of mode (D). In other words, we have the following possible transitions.

$$\begin{array}{ccccc} (A) & \overset{\longleftarrow}{\longrightarrow} & (B) & \longrightarrow & (D) \\ & \searrow & \uparrow & & \\ & & (C) & & \end{array}$$

We adopt the convention that the angle $\angle M$ formed by a straight line M with the horizontal axis belongs to $[0, \pi)$.

Now we will specify the set $Q \subset \widetilde{D} \times \widetilde{D}$ of all possible starting positions for the pair of processes (X_t, Y_t) and the corresponding starting modes. If $|\rho(x) - \rho(y)| \geq 7$ then $(x, y) \in Q$ and the starting mode is (A). If $|\rho(x) - \rho(y)| = 6$ and $x, y \in \widetilde{F}_{15}$ then we also have $(x, y) \in Q$ and we assume that if $(X_0, Y_0) = (x, y)$ then the processes start to move in mode (C). Let $M^{x,y}$ be the line of symmetry of the points x and y with the convention that the line is vertical if $x = y$. The starting mode is (B) and $(x, y) \in Q$ if $\angle M^{x,y} \in [\pi/3, 5\pi/6]$, $|\rho(x) - \rho(y)| < 7$, and $M^{x,y} \cap \widetilde{D} \subset \widetilde{F}_{10}$ but it is not true that $|\rho(x) - \rho(y)| = 6$ and $x, y \in \widetilde{F}_{15}$ (in that case, the starting mode is (C)). The diagonal of $\widetilde{D} \times \widetilde{D}$ is also a subset of Q. If the processes X_t and Y_t start from the same point, they will stay in mode (D) forever.

It is clear that each individual particle X_t and Y_t moves like a fiber Brownian motion on any interval which has only a finite number of transitions between the modes (A)-(D). We will show that there is only a finite number of transitions on any finite time interval. Note that in any sequence of mode transitions, mode (B) is visited over and over again, with the only excursions from this mode taking the form

$$(B) \longrightarrow (A) \longrightarrow (B),$$

$$(B) \longrightarrow (A) \longrightarrow (C) \longrightarrow (B),$$

or the only infinite excursion

$$(B) \longrightarrow (D).$$

It is clear that there exist $t_0, p_0 > 0$ such that for any points x and y satisfying $|\rho(x) - \rho(y)| \geq 7$, and the processes (X_t, Y_t) starting from (x, y), the first time t with the property $|\rho(X_t) - \rho(Y_t)| = 6$ will be greater than t_0 with probability greater than p_0. Hence, the processes will spend at least t_0 units of time in mode (A) with probability greater than p_0, each time they enter that mode. A standard argument now shows that there will be only a finite number of transitions $(A) \longrightarrow (B)$ on any finite time interval.

We will argue that if the processes X_t and Y_t start from a point $(X_0, Y_0) \in Q$, then they will never leave the set Q. As a part of this argument, we will show that if the processes start from a point in Q, they will never hit F while being in mode (B). Such an event would cause the particles to stop, according to the definition of mode (B). We included such a possibility in the definition of mode (B) so that we would not have to explain what a mirror coupling does when one of the particles enters a fiber γ_x^1.

Suppose that the particles start from an initial position in Q, in mode (A). Hence, they start from (x, y) such that $|\rho(x) - \rho(y)| \geq 7$ and they stay in this mode until time s defined by T^A, i.e., such that $|\rho(X_s) - \rho(Y_s)| = 6$. Until that time, $|\rho(X_t) - \rho(Y_t)| \geq 6$, by the continuity of paths of X and Y. All points (x, y) with $|\rho(x) - \rho(y)| \geq 6$ belong to Q so the process (X_t, Y_t) stays in Q as long as it is in mode (A).

Assume now that the processes start from $x, y \in \widetilde{F}_{15}$ with $|\rho(x) - \rho(y)| = 6$, in mode (C). According to (C) and T^C, as long as the processes stay in mode (C), we have $|\rho(X_t) - \rho(Y_t)| = 6$ and $X_t, Y_t \in \widetilde{F}_{15}$. Thus, the processes stay in Q when they are in mode (C).

The case of mode (D) is trivial so it remains to discuss mode (B). Suppose that the processes start from a position in Q corresponding to (B). Without loss of generality we may assume that the particles are initially in $\widetilde{F}_{10} \cap D_L$. Note that the initial position M_0 of the mirror is such that $\rho(z) \leq -4 - \alpha/2$ for all $z \in M_0 \cap \widetilde{D}$. As long as none of the particles touches F, the angle $\angle M_t$ of the mirror stays within $[\pi/3, 5\pi/6]$ by the results of Bañuelos and Burdzy (1999), and it is easy to see that the particles remain in Q. We will argue that none of the particles can hit F if they are in mode (B). Suppose otherwise. Let t_0 be the first time one of the particles hits F. Note that $|\rho(X_{t_0}) - \rho(Y_{t_0})| \leq 7$ and so $M_{t_0} \cap \widetilde{D}$ is not a subset of $\widetilde{F}_{10}$. Let t_1 be the last time before t_0 such that $M_{t_1} \cap \widetilde{D} \subset \widetilde{F}_{10}$. Since $|\rho(X_{t_1}) - \rho(Y_{t_1})| \leq 7$, for some $t_2 \in (t_1, t_0)$, both particles stay in $D_L^1 \cap \widetilde{F}_{15}$ in the interval (t_1, t_2). However, as long as both processes stay in $(D_L^1 \cap \widetilde{F}_{15}) \setminus F$, the interval $J_t = \{a : \rho(z) = a \text{ for some } z \in M_t \cap \widetilde{D}\}$ can only decrease by the results of Bañuelos and Burdzy (1999). This contradicts the definition of t_1 and proves our claim.

Next we introduce a partial order $\prec$ on $D_L \setminus F$. We start by considering points $x, y \in D_L^1$. We say that $x \prec y$ if and only if $\angle M^{x,y} \in [\pi/3, 5\pi/6]$ and $\rho(x) < \rho(y)$. We extend the definition to $v, z \in D_L \setminus F$ by declaring that $v \prec z$ if and only if for some $x, y \in D_L^1$ we have $x \prec y$, $v \in \mathcal{T}x$ and $z \in \mathcal{T}y$.

Lemma 5.1. *Let $u(t,x)$ be the Neumann heat equation solution in D with the initial condition $f(x)$, so that $u(t,x) = \mathbb{E}^x f(X_t)$, where the expectation is taken with respect to fiber Brownian motion in D. Consider the initial condition $f(x) = 1$ for x to the left of $\mathcal{L}$ and $f(x) = -1$ for all other x. The corresponding solution is monotone relative to $\prec$ in the sense that for all $x, y \in D_L \setminus F$ such that $x \prec y$, we have $u(t,x) \geq u(t,y)$ for all $t \geq 0$.*

Proof. Consider any $x, y \in D_L \setminus F$ with $x \prec y$. Find $v, z \in \widetilde{D}$ such that $x \in \mathcal{T}v$ and $y \in \mathcal{T}z$. Run a process (X_t, Y_t) in $\widetilde{D} \times \widetilde{D}$, starting from (v, z), according to the coupling recipe described in this section.

The point of the complicated coupling construction was that $\rho(X_t) \leq \rho(Y_t)$ for all $t \geq 0$, with probability one. Since $f(x_1) \geq f(x_2)$ whenever $\rho(x_1) \leq \rho(x_2)$, this implies that

$$\mathbb{E}^v f(X_t) \geq \mathbb{E}^z f(Y_t).$$

With a slight abuse of notation, for any $x_1 \in D_L$, let $\mathcal{T}^{-1}x_1$ denote the unique point x_2 in $\widetilde{D}$ with the property that $x_1 \in \mathcal{T}x_2$. For $x_1 \in D_R$ we let $\mathcal{T}^{-1}x_1 = \mathcal{L}\mathcal{T}^{-1}\mathcal{L}x_1$. Finally, we extend the mapping $\mathcal{T}^{-1}$ to all points in D in such a way that (i) every continuous path in D is mapped onto a continuous path in $\widetilde{D}$, and (ii) for every $x_1 \in D$ we have $\rho(\mathcal{T}^{-1}x_1) = \rho(x_1)$. Note that $f(\mathcal{T}^{-1}x_1) = f(x_1)$ for every $x_1 \in D$.

Let X_t^* be a fiber Brownian motion in D with $X_0^* = x$. It is evident that the process $\mathcal{T}^{-1}X_t^*$ is a fiber Brownian motion in $\widetilde{D}$ and so it has the same distribution as X_t, starting from $X_0 = v = \mathcal{T}^{-1}x$. Since $f(\mathcal{T}^{-1}X_t^*) = f(X_t^*)$ for every t,

$$\mathbb{E}^x f(X_t^*) = \mathbb{E}^x f(\mathcal{T}^{-1}X_t^*) = \mathbb{E}^v f(X_t).$$

If Y_t^* denotes a fiber Brownian motion in D starting from $Y_0^* = y$, we obtain in an analogous way,

$$\mathbb{E}^y f(Y_t^*) = \mathbb{E}^y f(\mathcal{T}^{-1}Y_t^*) = \mathbb{E}^z f(Y_t).$$

Hence,

$$u(t,x) = \mathbb{E}^x f(X_t^*) = \mathbb{E}^v f(X_t) \geq \mathbb{E}^z f(Y_t) = \mathbb{E}^y f(Y_t^*) = u(t,y).$$

□

6. Eigenvalue and eigenfunction estimates.

In the first few lemmas of this section, we will write D' to denote a Lipschitz domain such that $D' \cap F^c = F^c$, where F^c denotes $\mathbb{R}^2 \setminus F$.

Lemma 6.1. *Suppose D' is a Lipschitz domain whose intersection with F^c agrees with F^c and let $p_t(x,y)$ be the transition density for Brownian motion reflected in D'. There exists c_1 depending on t_0 but not on the shape of $D' \cap F$, such that $p_{t_0}(x,x) \le c_1$ for $x \in F_2^c$.*

Proof. Let $G = D \cap (\partial F_1)$ and $H = D \cap (\partial F_{7/4})$. Let $T_G = \inf\{t : X_t \in G\}$, and define T_H similarly.

Fix some $t_0 > 0$. The set $F_{3/2} \setminus F_{1/2}$ is a Lipschitz domain so, by the proof of Theorem 3.5 in Bass and Hsu (1991) and Remark 3.1, there exists $c_2 > 0$ such that the process started at $z \in G$ will move at least 1/8 away from the boundary before hitting $\partial F_{3/2} \cup \partial F_{1/2}$, with probability greater then c_2. This, the strong Markov property and the support theorem for Brownian motion (Theorem I.6.6 of Bass (1995)) imply that there exists $p_1 < 1$ such that $\mathbb{P}^z(T_H < t_0) < p_1$ for all $z \in G$.

Let $\overline{p}_t(x,y)$ be the transition density for reflecting Brownian motion in $D' \setminus F_{1/2}$ with absorption on G. To get from x to y, the paths either go from x to y without hitting G or else they first hit G. So for $x, y \in F_{1/2}^c$,

$$p_t(y,x) = \overline{p}_t(y,x) + \int_0^t \int_G \mathbb{P}^y(X_{T_G} \in dz, T_G \in ds) p_{t-s}(z,x).$$

If $t \le t_0$, $x \in F_2^c$ and $y \in F_{1/2}^c$ we have

$$\begin{aligned} p_t(y,x) &\le \overline{p}_t(y,x) + \Big(\sup_{0 \le r \le t} \sup_{z \in G} p_r(z,x) \Big) \int_0^t \int_G \mathbb{P}^y(X_{T_G} \in dz, T_G \in ds) \\ &\le \overline{p}_t(y,x) + \sup_{0 \le r \le t} \sup_{z \in G} p_r(z,x). \end{aligned} \tag{6.1}$$

For $z \in G$ and $x \in F_2^c$,

$$\begin{aligned} p_r(z,x) &= \int_0^r \int_H \mathbb{P}^z(X_{T_H} \in dw, T_H \in ds) p_{r-s}(w,x) \\ &\le \Big(\sup_{0 \le u \le r} \sup_{w \in H} p_u(w,x) \Big) \mathbb{P}^z(T_H < r). \end{aligned} \tag{6.2}$$

Taking the supremum over $t \le t_0$ and $y \in H$ in (6.1) and applying (6.2) we obtain for $x \in F_2^c$,

$$\begin{aligned} \sup_{t \le t_0} \sup_{y \in H} p_t(y,x) &\le \sup_{t \le t_0} \sup_{y \in H} \overline{p}_t(y,x) + \sup_{0 \le r \le t_0} \sup_{z \in G} p_r(z,x) \\ &\le \sup_{t \le t_0} \sup_{y \in H} \overline{p}_t(y,x) + \Big(\sup_{t \le t_0} \sup_{w \in H} p_t(w,x) \Big) \sup_{z \in G} \mathbb{P}^z(T_H < t_0). \end{aligned}$$

Since $\overline{p}_t(x,y) \le c_2 t^{-1}\exp(-c_3|x-y|^2/t)$ by Theorem 3.1 of Bass and Hsu (1991), we obtain

$$\sup_{t\le t_0}\sup_{y\in H}\sup_{x\in F_2^c}\overline{p}_t(y,x)\le c_4,$$

and, therefore, recalling that $\mathbb{P}^z(T_H<t_0)<p_1$ for $z\in G$,

$$\sup_{t\le t_0}\sup_{y\in H}p_t(y,x)\le c_4+p_1\Big(\sup_{t\le t_0}\sup_{y\in H}p_t(y,x)\Big),$$

or

$$\sup_{t\le t_0}\sup_{y\in H}p_t(y,x)\le \frac{c_4}{1-p_1}=c_5,$$

for $x\in F_2^c$. By (6.2), for $x\in F_2^c$,

$$\sup_{0\le r\le t_0}\sup_{z\in G}p_r(z,x)\le \sup_{0\le r\le t_0}\sup_{w\in H}p_r(w,x).$$

We apply this inequality together with (6.1) , but this time with $y=x\in F_2^c$, to obtain

$$\begin{aligned}p_{t_0}(x,x)&\le \overline{p}_{t_0}(x,x)+\sup_{0\le r\le t_0}\sup_{z\in G}p_r(z,x)\\&\le c_6+\sup_{0\le r\le t_0}\sup_{w\in H}p_r(w,x)\le c_7.\end{aligned}$$

This completes the proof. □

Next we want to obtain a modulus of continuity result for λ-resolvents in $D'\setminus F_1$. We set $R^\lambda g(x)=\mathbb{E}^x\int_0^\infty e^{-\lambda t}g(X_t)\,dt$ for $\lambda>0$. We will obtain a modulus of continuity for $R^\lambda g$ depending only on λ and the suprema of $|g|$ and $|R^\lambda g|$ on $D'\setminus F_1$ and not the shape of $D'\cap F_1$.

Lemma 6.2. *For every $\lambda>0$ there exist c_1 and ζ such that if $x,y\in D'\setminus F_1$, then*

$$|R^\lambda g(x)-R^\lambda g(y)|\le c_1|x-y|^\zeta\Big(\|g\|_{L^\infty(D'\setminus F_1)}+\|R^\lambda g\|_{L^\infty(D'\setminus F_1)}\Big).$$

Proof. It will suffice to fix arbitrarily small $\varepsilon>0$ and prove the lemma only for $x,y\in D'\setminus F_1$ whose distance apart is less than ε. Find an $\varepsilon>0$ and two disjoint small balls K_1 and K_2 in $D'\setminus F_1$ with the property that for every pair of points $x,y\in D'\setminus F_1$ with $|x-y|<\varepsilon$, the following is true for one of the balls, for example, ball K_1. We have $x\in D'\setminus(F_1\cup K_1)$ and if we let $\gamma=\operatorname{dist}(x,K_1)/4$ then both x and y are in the ball of radius $\gamma/4$ about x.

We will now fix one of the balls, K_1 say, and prove the lemma for all x and y satisfying the above condition. The lemma can be proved for all other pairs x and y by repeating the argument for K_2 in place of K_1.

Let T_{K_1} be the hitting time of K_1 and $R_1^\lambda g(x) = \mathbb{E}^x \int_0^{T_{K_1}} e^{-\lambda t} g(X_t)\,dt$ for $\lambda \geq 0$. Suppose h is a bounded function. We have

$$R_1^0 h(x) = \mathbb{E}^x \int_0^{T_\gamma} h(X_t)\,dt + \mathbb{E}^x R_1^0 h(X_{T_\gamma}),$$

where $T_\gamma = \inf\{t : X_t \notin B(x,\gamma)\}$, and similarly for $R_1^0 h(y)$. Then

$$|R_1^0 h(x) - R_1^0 h(y)| \leq 2\|h\|_{L^\infty(D'\setminus F_1)} \sup_{z \in B(0,\gamma)} \mathbb{E}^z T_\gamma + |\mathbb{E}^x R_1^0 h(X_{T_\gamma}) - \mathbb{E}^y R_1^0 h(X_{T_\gamma})|.$$

Since the amount of time for a reflecting Brownian motion in a Lipschitz domain to leave a ball of radius γ is bounded by a constant times γ^2 (cf. Bass and Hsu (1991), Corollary 3.3), then the first term on the right is bounded by $c_2\gamma^2\|h\|_{L^\infty(D'\setminus F_1)}$. The function $z \to \mathbb{E}^z R_1^0 h(X_{T_\gamma})$ is harmonic with respect to reflecting Brownian motion, so by Bass and Hsu (1991) Corollary 3.8 and scaling, there exists $\zeta_1 \in (0,1)$ such that the second term on the right hand side is bounded by

$$c_3 \frac{|x-y|^{\zeta_1}}{\gamma^{\zeta_1}} \|R_1^0 h\|_{L^\infty(D'\setminus F_1)}.$$

Choosing $\gamma = |x-y|^{\zeta_1/(2+\zeta_1)}$ and adding, we see that

$$|R_1^0 h(x) - R_1^0 h(y)| \leq c_4 |x-y|^\zeta \Big(\|h\|_{L^\infty(D'\setminus F_1)} + \|R_1^0 h\|_{L^\infty(D'\setminus F_1)} \Big), \tag{6.3}$$

where $\zeta = 2\zeta_1/(2+\zeta_1)$.

Now let g be bounded and let $h = -\frac{1}{2}\Delta R_1^\lambda g$ so that $R_1^0 h = R_1^\lambda g$. Since $(\lambda - \frac{1}{2}\Delta) R_1^\lambda$ is the identity, then $h = g - \lambda R_1^\lambda g$. So both h and $R_1^0 h$ are bounded in L^∞ norm by

$$(1+\lambda)\Big(\|g\|_{L^\infty(D'\setminus F_1)} + \|R_1^\lambda g\|_{L^\infty(D'\setminus F_1)} \Big).$$

Applying (6.3) to $R_1^0 h = R_1^\lambda g$, we obtain our estimate. □

Recall that D' denotes a Lipschitz domain. Such a domain has a discrete spectrum for the half Laplacian with Neumann boundary conditions. Let μ_k be the sequence of all eigenvalues, ordered and repeated, if necessary. Let φ_k be the corresponding eigenfunctions. Note that $\mu_1 = 0$ and φ_1 is a constant function.

Corollary 6.3. *If $\mu_2 < c_1$ then one can find a modulus of continuity for φ_2 in $D' \setminus F_2$ which does not depend on D' but only c_1.*

Proof. By Lemma 6.1,

$$c_2 \geq p_1(x,x) = \sum_i e^{-\mu_i} \varphi_i(x)\varphi_i(x),$$

so

$$\varphi_2(x)^2 \le c_2 e^{\mu_2} \le c_3.$$

Now $\varphi_2 = (1+\mu_2)R^1\varphi_2$, since φ_2 is an eigenfunction. So on $D' \setminus F_2$ we have

$$|R^1\varphi_2(x)| = (1+\mu_2)^{-1}|\varphi_2(x)| \le c_3^{1/2}.$$

By Lemma 6.2, $R^1\varphi_2$ has a modulus of continuity not depending on D'. Using $\varphi_2 = (1+\mu_2)R^1\varphi_2$, we thus obtain a modulus of continuity for φ_2. □

Recall that D' is a Lipschitz domain whose intersection with F^c agrees with F^c. Let f be a bounded function, and let u be the solution to the heat equation in D' with initial condition $u(0,x) = f(x)$. The solution is given by

$$u(t,x) = \int p_t(x,y) f(y)\, dy,$$

where $p_t(x,y)$ are transition densities for the reflected Brownian motion in D'. The eigenfunction expansion of $p_t(x,y)$ yields

$$u(t,x) = \sum_i a_i e^{-\mu_i t}\varphi_i(x),$$

where $a_i = \int f(y)\varphi_i(y)\, dy$.

Lemma 6.4. *Suppose D' is as above and there exist b_1, b_2 such that $\mu_2 < b_1 < b_2 < \mu_3$. There exists $\gamma > 0$ and c_1 depending only on b_1, b_2 such that for all $x \in D' \setminus F_1$ and $t \ge 1$,*

$$|u(t,x) - a_1 - a_2 e^{-\mu_2 t}\varphi_2(x)| \le c_1 \|f\|_2 e^{-(\mu_2+\gamma)t}.$$

Proof. By the Cauchy-Schwarz inequality,

$$|u(t,x) - a_1 - a_2 e^{-\mu_2 t}\varphi_2(x)| = \Big|\sum_{i>2} a_i e^{-\mu_i t}\varphi_i(x)\Big| \le \Big(\sum_{i>2} a_i^2\Big)^{1/2} \Big(\sum_{i>2} e^{-2\mu_i t}\varphi_i(x)^2\Big)^{1/2}.$$

By Bessel's inequality, $\sum_i a_i^2 \le \int f^2$. We take $\gamma = (\mu_3 - \mu_2)/2$. Using Lemma 6.1 and the fact that $\mu_3 > b_2 > b_1 > \mu_2$, we obtain,

$$\begin{aligned}
\sum_{i>2} e^{-2\mu_i t}\varphi_i(x)^2 &\le e^{-(2\mu_2+2\gamma)t} \sum_{i>2} e^{-2\gamma t}\varphi_i(x)^2 \\
&\le e^{-(2\mu_2+2\gamma)t} \sum_{i>2} e^{-2\gamma}\varphi_i(x)^2 \\
&\le e^{-(2\mu_2+2\gamma)t} p_{2\gamma}(x,x) \le c_2 e^{-(2\mu_2+2\gamma)t}.
\end{aligned}$$

Hence,

$$|u(t,x) - a_1 - a_2 e^{-\mu_2 t}\varphi_2(x)| \le \Big(\int f^2\Big)^{1/2} \Big(c_2 e^{-(2\mu_2+2\gamma)t}\Big)^{1/2} = c_2^{1/2}\|f\|_2 e^{-(\mu_2+\gamma)t}.$$

This completes the proof. □

Lemma 6.5. *Suppose that $u(t,x)$ is the solution of the heat equation in D' with $u(0,x) = f(x)$, where $\|f\|_2 < c_1$. For a fixed $t > 0$, one can find a modulus of continuity for $x \to u(t,x)$ on $D \setminus F_2$ which depends only on c_1 and t.*

Proof. Fix some $\lambda > 0$ and let

$$h(x) = \sum_i a_i(\lambda + \mu_i)e^{-\mu_i t}\varphi_i(x),$$

where $a_i = \int f(y)\varphi_i(y)dy$. Since the φ_i are eigenfunctions, $R^\lambda \varphi_i = (\lambda + \mu_i)^{-1}\varphi_i$, and hence $u(t,x) = R^\lambda h(x)$. Here R^λ is the λ-resolvent.

Note that $(\lambda + \mu)^2 e^{-\mu t}$ is bounded for $\mu \ge 0$ by a constant depending only on t. This implies that

$$\sum_i (\lambda + \mu_i)^2 e^{-2\mu_i t}\varphi_i(x)^2 \le c_2 \sum_i e^{-\mu_i t}\varphi_i(x)^2 = c_2 p_t(x,x).$$

By Lemma 6.1, $p_t(x,x)$ is bounded on $D' \setminus F_2$ by c_3. By Bessel's inequality, $\sum_i a_i^2 \le \int f^2 \le c_1^2$. Putting these estimates together and using the Cauchy-Schwarz inequality,

$$|h(x)| \le \Big(\sum_i (\lambda + \mu_i)^2 e^{-2\mu_i t}\varphi_i(x)^2\Big)^{1/2} \Big(\sum_i a_i^2\Big)^{1/2} \le c_2 c_3 c_1.$$

By Lemma 6.2, $u(t,x) = R^\lambda h(x)$ is continuous on $D \setminus F_2$ with a modulus dependent only on c_1 and t. □

Lemma 6.6. *For every open set M and reals $a_1, a_2, b_1, b_2 > 0$, there exists $\varepsilon > 0$ with the following property. If φ satisfies $\frac{1}{2}\Delta\varphi = -\mu\varphi$ in M with $b_1 < \mu < b_2$, and $a_1 < \varphi(x) < a_2$ then $a_2 > a_1 + \varepsilon$.*

Proof. Pick a point $x_0 \in M$ and let $\gamma = \mathrm{dist}\,(x_0, \partial M)/2$. Let T be the first time the Brownian motion starting from x_0 leaves $B(x_0,\gamma)$ and suppose $x \in B(x_0,\gamma)$. Since $\frac{1}{2}\Delta\varphi = -\mu\varphi$, then by Itô's formula,

$$\varphi(x) = \mathbb{E}^x \varphi(X_T) + \mu \mathbb{E}^x \int_0^T \varphi(X_s)ds. \tag{6.4}$$

Now suppose $a_1 < \varphi < a_2$ on M and $a_2 - a_1 = \varepsilon$ is small. The oscillation of the left hand side of (6.4) over the set $B(x_0, \gamma)$ is less than ε, i,.e.,

$$\sup_{x \in B(x_0,\gamma)} \varphi(x) - \inf_{x \in B(x_0,\gamma)} \varphi(x) \leq \varepsilon.$$

The oscillation of the first term on the right of (6.4) is also less than ε. We write the second term on the right of (6.4) as

$$\mu \mathbb{E}^x \int_0^T (\varphi - a_1)(X_s) ds + a_1 \mu \mathbb{E}^x T.$$

The first term has oscillation less than

$$\mu \varepsilon \mathbb{E}^x T \leq \mu \varepsilon c_1 \gamma^2 \leq c_1 b_2 \varepsilon \gamma^2.$$

But the oscillation of the second term is greater than $a_1 \mu$ times the oscillation of $\mathbb{E}^x T$, hence is bigger than $c_1 a_1 b_1 \gamma^2$. Hence, the oscillation of the right hand side of (6.4) is larger than $c_1 a_1 b_1 \gamma^2 - c_1 b_2 \varepsilon \gamma^2 - \varepsilon$ and this is larger than the oscillation of the left hand side, ε, if ε is sufficiently small. This is a contradiction. The proof is complete. □

Recall the set D and various elements of its construction from Section 5. By the results of Sections 2 and 3, there exists a decreasing sequence of open sets $\widetilde{D}_n$ such that reflected Brownian motions in $\widetilde{D}_n$'s converge to the fiber Brownian motion in D. Every domain $\widetilde{D}_n$ is obtained by removing a finite number of line segments and polygons from D. One can approximate every domain $\widetilde{D}_n$ by a sequence of Lipschitz domains such that reflected Brownian motions in those domains converge to the reflected Brownian motion in $\widetilde{D}_n$. Finding such a sequence of Lipschitz domains is rather easy in comparison with the construction presented in Sections 2 and 3 so we will leave the details to the reader. Using the diagonal method, we see that there exists a sequence of Lipschitz domains D_n such that reflected Brownian motions in D_n's converge to the fiber Brownian motion in D. Moreover, we can construct D_n's so that $D_n \cap F^c = F^c$ for every n.

Let $\{\mu_k^n\}_{k \geq 1}$ be the sequence of Neumann eigenvalues for D_n, ordered according to the size and repeated, if necessary. The corresponding eigenfunctions will be denoted φ_k^n. Recall that $\mathbf{w}$ is the width of the fiber bundles connecting the two large subdomains of D.

Lemma 6.7 *We can find $\mathbf{w} \in (0, 1/100)$ and constants $c_1 < c_2 < c_3$ such that for large n,*

$$c_1 < \mu_2^n < c_2 < c_3 < \mu_3^n.$$

Proof. We will first prove that for any $c_2 > 0$ we can have $\mu_2^n < c_2$, by choosing sufficiently small $\mathbf{w} > 0$. The proof is almost identical to that of Lemma 1 in Burdzy and Werner (1999).

Let A_5 be the point at the intersection of the line containing A_1 and A_2 and the horizontal axis. For $x \in D_L^1$, define a function $f(x)$ as follows,

(i) $f(x) = 0$ if $|x| > 5$ or if $\operatorname{dist}(x, A_5) < 10\mathbf{w}$,

(ii) $f(x) = \log(\operatorname{dist}(x, A_5)/10\mathbf{w})/\log(1/10\mathbf{w})$ if $|x| \le 5$ and $\operatorname{dist}(x, A_5) \in [10\mathbf{w}, 1]$,

(iii) $f(x) = 1$ if $|x| < 5$ and $\operatorname{dist}(x, A_5) > 1$.

We extend f into a continuous function on D_n which is constant on every set $\mathcal{T}x \subset D_L$, which is antisymmetric with respect to $\mathcal{L}$, i.e., $f(x) = -f(\mathcal{L}x)$ for $x \in D_L$, and which vanishes outside $D_L \cup D_R$. The function f satisfies the Neumann boundary conditions on ∂D_n. It is easy to check that when $\mathbf{w} \to 0$, the integrals (over D_n), $\int |f|$ and $\int |f|^2$ remain bounded above and bounded away from 0, and $\int |\nabla f|^2 \to 0$ when $\mathbf{w} \to 0$. Since the function f is orthogonal to the constant function 1 (i.e., to the lowest eigenfunction), we have

$$\mu_2^n \le \frac{\int |\nabla f|^2}{\int |f|^2}.$$

In view of the previous remarks, the quantity on the right hand side can be made smaller than c_2 by choosing small $\mathbf{w}$. Note that the estimate does not depend on n.

Next we will prove that the second Neumann eigenvalue μ_2^n in D_n is simple if $\mathbf{w}$ is small.

It is easy to see that there exists $\xi < 10^{-3}$ with the following property. Suppose that U is a subdomain of F_{16}^c whose diameter is less than ξ and consider Brownian motion in U, reflected on $\partial U \cap \partial D$ and killed on $\partial U \setminus \partial D$. Then with probability greater than 1/2, the Brownian motion in U is killed within the first time unit of its motion, independent of the starting point. This estimate is independent of $\mathbf{w} < 1/100$ and of the shape of U. The estimate implies that the probability of survival for more than t units of time is less than $(1/2)^t$, for integer $t > 0$. This implies that the first mixed eigenvalue in U, with Neumann boundary conditions on $\partial U \cap \partial D$ and Dirichlet conditions on $\partial U \setminus \partial D$, must be greater than $c_4 = (\log 2)/2$. Now assume that $\mathbf{w}$ is so small that $c_2 < c_4$. The nodal line of any second eigenfunction φ_2^n divides D_n into two subdomains by the Courant nodal line theorem (Bandle (1980), p. 112 or Chavel (1984), p. 19). None of those subdomains can lie inside F_{16}^c and have diameter less than ξ because φ_2^n truncated to such a subdomain U is the first mixed eigenfunction in U with eigenvalue μ_2^n. This eigenvalue would have to be greater than c_4 and less than c_2, a contradiction.

Consider any second eigenfunction φ_2^n and the function $\widetilde{\varphi}_2^n(x) = \varphi_2^n(x) - \varphi_2^n(\mathcal{L}x)$. If φ_2^n is not $\mathcal{L}$-symmetric then $\widetilde{\varphi}_2^n$ is a non-zero antisymmetric eigenfunction, i.e., such that $\widetilde{\varphi}_2^n(x) = -\widetilde{\varphi}_2^n(\mathcal{L}x)$. Hence, there exists either a symmetric or antisymmetric eigenfunction.

In either case, there exists an eigenfunction whose nodal line is $\mathcal{L}$-symmetric. We will consider such an eigenfunction in the next paragraph.

We will show that the nodal line Γ of φ_2^n cannot intersect F_{17}^c. In view of what we have just proved, if Γ intersects F_{17}^c, then the diameter of $\Gamma \cap F_{16}^c$ must be greater than ξ. For a fixed $\xi > 0$, there is a $p > 0$ such that for every starting point $x \in D_n$, for large n, the reflected Brownian motion in D_n will hit either $\Gamma \cap F_{16}^c$ or $\mathcal{L}(\Gamma \cap F_{16}^c)$ within the first unit of time with probability p. This follows from the fact that the reflected Brownian motions in D_n converge weakly to a fiber Brownian motion in D and the motion along the tubes connecting D_L and D_R has the character of a one-dimensional Brownian motion. The same estimate applies to the process in D_n. Hence, the probability of not hitting the nodal line of φ_2^n within the first t units of time for integer $t > 0$ is less than $(1-p)^t$. This translates to the following statement about eigenvalues. The first mixed eigenvalue in a nodal domain of φ_2^n must be greater than $c_5 = -(\log(1-p))/2$. In other words, $\mu_2^n > c_5$, where c_5 does not depend on $\mathbf{w}$. By choosing sufficiently small $\mathbf{w}$ we would have $\mu_2^n < c_2 < c_5/2$. This would be a contradiction, which shows that for small $\mathbf{w}$, the nodal line of φ_2^n does not intersect F_{17}^c. By Lemma 3 of Burdzy and Werner (1999), μ_2^n is simple, i.e., φ_2^n is unique up to a multiplicative constant.

By uniqueness and the analysis above, the eigenfunction φ_2^n must be either symmetric or anti-symmetric. Suppose first that it is symmetric. One of its two nodal subdomains cannot intersect F_{17}^c, since the nodal line Γ cannot do that. An argument completely analogous to the ones given above shows that the first mixed eigenvalue in such a subdomain must be larger than c_2, which is a contradiction. We conclude that φ_2^n is antisymmetric and so $\Gamma_1 = \mathcal{L} \cap D_n$ is a part of the nodal line. However, the nodal line of the second eigenfunction divides the domain into only two subdomains, so Γ_1 is in fact the whole nodal set of φ_2^n.

For a fixed $\mathbf{w} > 0$ there exists $p_1 > 0$ such that for large n and all $x \in D_n$, reflected Brownian motion in D_n can hit Γ_1 within one unit of time with probability greater than p_1. As before, this translates into a lower bound for the first mixed eigenvalue for the domain to the left of Γ_1 and this in turn gives a lower bound for the second eigenvalue μ_2^n. This completes the proof of the claim that $\mu_2^n > c_1$.

Next we turn to the analysis of φ_3^n. By the same argument as in the case of φ_2^n, there exists either a symmetric or anti-symmetric third eigenfunction. Suppose that φ_3^n is antisymmetric. Then Γ_1 is a part of the nodal set for φ_3^n and the number of nodal subdomains must be even. The number must not exceed three so there must be two nodal subdomains. Hence, Γ_1 is the whole nodal set of φ_3^n. This and the antisymmetry of both φ_3^n and φ_2^n imply that $\int \varphi_2^n \varphi_3^n \neq 0$. This is a contradiction, proving that there is no anti-symmetric third eigenfunction. It follows that there exists a symmetric third eigenfunction.

Our earlier arguments show that if the nodal line of φ_3^n intersects F_{17}^c, then the eigenvalue μ_3^n can be bounded below and we are done. If the nodal line does not intersect F_{17}^c, then there is a nodal subdomain which is disjoint from F_{17}^c. The usual arguments show that the Brownian motion in such a subdomain, with reflection on ∂D_n and killing on the rest of the boundary, has to be killed fairly quickly and this leads to a lower bound on the first mixed eigenvalue, independent of $\mathbf{w}$. This completes the proof of $\mu_3^n > c_3 > c_2$. $\square$

Remark 6.8. It is not hard to see that the domains D_n may be constructed in such a way that for any fixed $a, b, c \in [-15 - \alpha/2, 15 + \alpha/2]$ such that $a < b < c$, the probability that the reflected Brownian motion in D_n starting from any point $x \in D_n$ with $\rho(x) = b$ will hit the set $\{y \in D_n : \rho(y) = a\}$ before hitting $\{y \in D_n : \rho(y) = c\}$ converges to $(c-b)/(c-a)$ as $n \to \infty$, uniformly in $x \in D_n$ with $\rho(x) = b$.

Lemma 6.9. *There exists $c_1 > 1$ such that for sufficiently small $\mathbf{w}$ and large n,*

$$\sup_{x \in D_n \cap F_{13}^c} \varphi_2^n > c_1 \sup_{x \in D_n \cap F_{10}} \varphi_2^n.$$

Proof. Let G_n be the part of D_n between ∂F_{13} and $\mathcal{L}$. Choose t_1 and λ with the following property: the probability that reflected Brownian motion in G_n starting from any point of ∂F_{10} will hit ∂F_{13} for the first time after time t but before hitting $\mathcal{L}$ is less than $\exp(-\lambda t)$, for all $t \geq t_1$. Recall that α denotes the length of a fiber γ_x^k in D. We choose t_1 so large that

$$\sum_{k=0}^{\infty} \exp(-\lambda(t_1 + k)) \exp(\lambda(t_1 + k + 1)/2) \leq c_2 = \frac{11.5 + \alpha/2}{12 + \alpha/2} - \frac{11 + \alpha/2}{12 + \alpha/2}.$$

Choose $\mathbf{w}$ sufficiently small so that the eigenvalue μ_2^n is so small that $\mu_2^n < \lambda/2$ and $\exp(\mu_2^n t_1) < (13 + \alpha/2)/(12 + \alpha/2)$.

Let $u(t, x) = \varphi_2^n(x) \exp(\mu_2^n t)$. This function is parabolic for reflected Brownian motion X_t in G_n, stopped at the hitting time of $\mathcal{L}$ or ∂F_{13}. The averaging property of parabolic functions with respect to the process says that the value of $\varphi_2^n(x) = u(0, x)$ for any $x \in F_{10}$ can be computed as $\mathbb{E}^x u(\tau, X_\tau)$ where τ is the first hitting time of $\mathcal{L} \cup \partial F_{13}$. Note that $u(t, x) = 0$ for $x \in \mathcal{L}$ so we will concentrate on ∂F_{13}. By Remark 6.8, for sufficiently large n and all $x \in D_n \cap F_{10}$ we have

$$\mathbb{P}^x(X_\tau \in \partial F_{13}) \leq \frac{11 + \alpha/2}{13 + \alpha/2},$$

so

$$\begin{aligned}\mathbb{E}^x u(\tau, X_\tau)\mathbf{1}_{\{\tau<t_1\}} &\le \mathbb{P}^x(X_\tau \in \partial F_{13}) \sup_{y\in\partial F_{13}} \varphi_2^n(y)\exp(\mu_2^n t_1)\\ &\le \frac{11+\alpha/2}{13+\alpha/2} \sup_{y\in\partial F_{13}} \varphi_2^n(y)\cdot\frac{13+\alpha/2}{12+\alpha/2}\\ &\le \frac{11+\alpha/2}{12+\alpha/2} \sup_{y\in\partial F_{13}} \varphi_2^n(y).\end{aligned}$$

The other contribution to the expectation comes from the hitting after time t_1. Recall that we have chosen t_1 and λ in such a way that the probability that reflected Brownian motion in G_n starting from any point of ∂F_{10} will hit ∂F_{13} for the first time after time t but before hitting $\mathcal{L}$ is less than $\exp(-\lambda t)$, for all $t \ge t_1$. Hence,

$$\begin{aligned}\mathbb{E}^x u(\tau, X_\tau)\mathbf{1}_{\{\tau\ge t_1\}} &\le \sum_{k=0}^{\infty} \sup_{y\in\partial F_{13}} \varphi_2^n(y) \sup_{t\le t_1+k+1} e^{\lambda t}\,\mathbb{P}(\tau\in[t_1+k, t_1+k+1])\\ &\le \sup_{y\in\partial F_{13}} \varphi_2^n(y)\sum_{k=0}^{\infty}\exp(-\lambda(t_1+k))\exp(\lambda(t_1+k+1)/2)\\ &\le \sup_{y\in\partial F_{13}} \varphi_2^n(y)\cdot c_2\\ &\le \sup_{y\in\partial F_{13}} \varphi_2^n(y)\cdot\left(\frac{11.5+\alpha/2}{12+\alpha/2}-\frac{11+\alpha/2}{12+\alpha/2}\right).\end{aligned}$$

The two estimates yield for $x \in D_n \cap F_{10}$,

$$\varphi_2^n(x) = \mathbb{E}^x u(\tau, X_\tau) \le \frac{11.5+\alpha/2}{12+\alpha/2} \sup_{y\in\partial F_{13}} \varphi_2^n(y).$$

This directly implies the inequality in the lemma. □

Proof of Theorem 1.2. Choose $c_1 > 1$, n and $\mathbf{w}$ as in Lemma 6.9, so that

$$\sup_{x\in D_n\cap F_{13}^c} \varphi_2^n > c_1 \sup_{x\in D_n\cap F_{10}} \varphi_2^n.$$

We will assume without loss of generality that the L^2-norm of φ_2^n is equal to 1. It follows that there exists $c_2 > 0$, such that for n and $\mathbf{w}$ as in Lemma 6.9, we must have $\varphi_2^n(x) > c_2$ for some $x \in F_{13}^c$.

Fix some $\mathbf{w}$ as above and then use Lemma 6.7 to find $c_3 > 0$ such that $\mu_2^n > c_3$ for sufficiently large n.

Recall from the proof of Lemma 6.7 that the φ_2^n are antisymmetric. If one of these eigenfunctions has its maximum and, by antisymmetry, its minimum, inside the domain D_n

then we are done. We will assume that the opposite is true and show how this assumption leads to a contradiction.

Indeed, let us assume that the maxima of φ_2^n occur at points x_n on the boundary of $F_{13}^c \cap D_L$. Recall from Section 5 the partial order $\prec$ among the points of D. Let M be the set of all points $x \in D$ which satisfy $x \prec x_n$ for every n. It is not hard to see that M contains a non-degenerate ball around the origin.

Lemma 6.7 gives us a lower and an upper bound for μ_2^n. Now use Lemma 6.6 to find an $\varepsilon > 0$ such that if $\varphi_2^n(x) \in [c_2/2, c_2/2 + \varepsilon_1]$ for all $x \in M$ then $\varepsilon_1 > \varepsilon$.

Let $u_n(t,x)$ be the solution to the Neumann heat problem in D_n with the initial temperature equal to 1 to the left of $\mathcal{L}$ and equal to -1 elsewhere. Note that the L^2-norm of the initial condition $u_n(0,x)$ is bounded by some c_4, independent of n. The bounds in Lemma 6.7 allow us to apply Lemma 6.4. We see that for some c_5 and γ,

$$|u_n(t,x) - a_1^n - a_2^n e^{-\mu_2^n t}\varphi_2^n(x)| \leq c_5 c_4 e^{-(\mu_2^n+\gamma)t} = c_6 e^{-(\mu_2^n+\gamma)t}, \tag{6.5}$$

for all $t \geq 1$ and $x \in F_{13}^c$.

Recall from Corollary 6.3 that the φ_2^n are equicontinuous on $F_{13}^c \cap D_L$. This and the fact that every φ_2^n has L^2-norm equal to 1, imply that

$$a_2^n = \int_{D_n} u_n(0,x)\varphi_2^n(x)dx = \int_{D_n} |\varphi_2^n(x)|dx > c_7 > 0,$$

where c_7 is independent of n. Let $t_0 > 1$ be so large that $c_6 e^{-\gamma t_0}/c_7 < \varepsilon/16$.

Recall that $u(t,x)$ denotes the Neumann heat equation solution in D, i.e., $u(t,x) = \mathbb{E}^x u(0,X_t)$, where the expectation is taken with respect to the fiber Brownian motion in D. We take the same initial condition as in the case of $u_n(t,x)$, i.e., the initial value $u(0,x)$ is equal to 1 to the left of $\mathcal{L}$ and equal to -1 on the remaining part of D.

By Lemma 6.5, passing to a subsequence if necessary, we have that $u_n(t_0,\cdot)$ converge to $u(t_0,\cdot)$, uniformly on M. Find sufficiently large n such that $|u_n(t_0,x) - u(t_0,x)| < c_7(\varepsilon/8)\exp(-\mu_2^n t_0)$ for $x \in M$. For any $x \in M$ we have $x \prec x_n$, so $u(t_0,x) \geq u(t_0,x_n)$ by Lemma 5.1. We have

$$\begin{aligned}
0 &\leq u_n(t_0,x_n) - u_n(t_0,x) \\
&\leq |u_n(t_0,x_n) - u(t_0,x_n)| + (u(t_0,x_n) - u(t_0,x)) + |u_n(t_0,x) - u(t_0,x)| \\
&\leq 2c_7(\varepsilon/8)\exp(-\mu_2^n t_0).
\end{aligned}$$

This and (6.5) imply that

$$\begin{aligned}
|\varphi_2^n(x_n) - \varphi_2^n(x)| &\leq 2c_6 e^{-(\mu_2^n+\gamma)t_0}/(a_2^n e^{-\mu_2^n t_0}) + 2c_7(\varepsilon/8)\exp(-\mu_2^n t_0)/(a_2^n e^{-\mu_2^n t_0}) \\
&\leq 2c_6 e^{-\gamma t_0}/c_7 + \varepsilon/4 \leq 3\varepsilon/8.
\end{aligned}$$

From this, we obtain $|\varphi_2^n(x) - \varphi_2^n(y)| \le 3\varepsilon/4$ for all $x, y \in M$. This, however, contradicts the definition of ε. □

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Richard F. Bass
Department of Mathematics
University of Connecticut
Storrs, CT 06269
e-mail: bass@math.uconn.edu

Krzysztof Burdzy
Department of Mathematics
University of Washington
Box 354350
Seattle, WA 98195-4350
e-mail: burdzy@math.washington.edu