A book review of *Some Aspects of Brownian Motion Part II: Some Recent Martingale Problems* by Marc Yor

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Few would argue that Brownian motion is one of the most important stochastic processes. Although interest in this area has continued for nearly a century, the subject is very far from being exhausted. Some of the most modern research on martingale and distributional features of Brownian motion is described in the book *Some Aspects of Brownian Motion, Part II: Some Recent Martingale Problems* by Marc Yor (Birkhäuser, Basel, 1997).

The author gave a series of lectures at ETH in the winters of 1991-92 and 1993-94. Volume I of this two volume series, *Some Aspects of Brownian Motion, Part I: Some Special Functionals* (Birkhäuser, Basel, 1992) represents the first half of these lectures, while the current volume represents the second half. We give a brief overview of the subject of this book.

The book begins with Chapter 10 on the principal values of Brownian local times. (Chapters 1–9 form volume I.) If B_t is a Brownian motion, the random function $x \to \int_0^t 1_{(-\infty,x]}(B_s)\,ds$ turns out to be differentiable in x. The derivative is called local time and is denoted ℓ_t^x. The local time is Hölder continuous in both the x and t variables, and it thus makes sense to talk about the value of the Hilbert transform of ℓ_t^x at a point. This value can also be written as $\lim_{\varepsilon\to 0}\int_0^t (B_s)^{-1} 1_{(|B_s|\geq\varepsilon)} ds$. Chapter 10 is concerned with the distribution of this and related random variables and connections with excursions of Bessel processes.

Chapter 11 begins with a characterization of the Riemann zeta function in terms of the hitting time of 1 by a three-dimensional Bessel process started from 0. It goes on to develop many amazing connections between the zeta function and stochastic processes.

Let $\mathcal{F}_t$ be the σ-field generated by a Brownian motion up to time t. If we add additional information to $\mathcal{F}_t$, for example, if we add the value of B_t at time 1, then questions arise as to whether B_t remains a semimartingale in the enlarged filtration, and what is the structure of B_t under this enlargement. What may seems like a rather abstract question has a number of fascinating applications, including Williams' path decomposition and

Pitman's theorem on three-dimensional Bessel processes. This is the subject of Chapter 12.

The Burkholder–Gundy inequalities relate the $p/2^{th}$ moment of T to the p^{th} moment of $\sup_{s\leq T} B_s$, where T is a stopping time. Chapter 13 concerns what happens when one permits T to be a more general random time. That one can say anything is surprising; that one can say a great deal is extremely surprising.

If $\mathcal{Z}$ denotes the random set $\{t : B_t = 0\}$, what are the martingales that are zero precisely on $\mathcal{Z}$? The resolution of this problem takes up Chapter 14. As an application, one can use these rsults to find a martingale that has the same local time at 0 as Brownian motion.

If B_t is a Brownian motion, one can write any L^2 random variable adapted to $\mathcal{F}_t$ as a sum of multiple stochastic integrals with respect to Brownian motion. Azèma's and Emery's work has shown that there are other nontrivial martingales besides Brownian motion and compensated Poisson processes that enjoy this property; see Chapter 15.

When studying the Ray-Knight property of Brownian local times, it is natural to consider the filtrations $\mathcal{E}_t^x = \sigma(B_s \wedge x; s \leq t)$. Thus one considers processes indexed by space as well as by time. Chapter 16 contains many interesting results concerning this and related filtrations.

Chapter 17 deals with what is known about characterizing the Brownian filtration in terms of what martingales it supports. There are a number of old problems in this area (many originated by the author) that have recently been solved, although there are plenty more to tempt the reader.

Finally, Chapter 18 contains complements to the material in volume I.

This book is not for the beginner, but for the more advanced researcher it is a delight. I was reminded a bit of Ramanujan's notebooks; here too there are a huge number of beautiful and unexpected relationships between seemingly disparate subjects. Unlike Ramanujan's notebooks, however, here for every result either a proof or a detailed reference is included. This book is warmly recommended.