Brownian Motion, Heat Kernels, and Harmonic Functions

RICHARD F. BASS*

Department of Mathematics, University of Washington
Seattle, WA 98195, USA

ABSTRACT. Although the boundary behavior of harmonic functions is an old subject (Fatou's theorem was proved in 1906), interesting results are still being obtained today. In this article we discuss some recent results concerning harmonic functions, heat kernels, and related topics that have been obtained using Brownian motion. In the following sections we will discuss the heat kernels for the Neumann Laplacian, the boundary Harnack principle, the Martin boundary, conditional lifetimes, and the conditional gauge theorem.

1 Heat kernels and reflecting Brownian motion

Let $p^D(t,x,y)$ denote the Neumann heat kernel for a domain D. This is the fundamental solution to the heat equation $\partial u/\partial t = (1/2)\Delta u$ with Neumann boundary conditions. (Having Neumann boundary conditions means that the normal derivative of u is 0 on the boundary of D.) $p^D(t,x,y)$ is also, of course, the transition density of reflecting Brownian motion in D.

What can one say about $p^D(t,x,y)$ as a function of the domain D? The heat kernel with Dirichlet boundary conditions (that is, $u=0$ on the boundary of D) is easily seen to decrease as the domain D becomes smaller. One might expect that for Neumann boundary conditions, $p^{D_1}(t,x,y)$ should be greater than $p^{D_2}(t,x,y)$ if $D_1 \subseteq D_2$. A small room should warm up faster than a large room. A little thought shows that one must assume D_1 and D_2 are convex. With this assumption, the above monotonicity holds if D_2 is a ball centered at x [13], if a sphere about x separates ∂D_1 and ∂D_2 [19], if $\bar{D}_1 \subseteq D_2$ and t is sufficiently small [12], or if t is sufficiently large [13].

It turns out, however, that this domain monotonicity need not always hold. Fix $t>0$. Let $\varepsilon > 0$ and set

$$D_2 = \{re^{i\theta} : r>0, 0<\theta<3\pi/4\}, \qquad D_1 = D_2 + (-\varepsilon, \varepsilon),$$

$$x_\varepsilon = (-\varepsilon,\varepsilon), \qquad y_\varepsilon = (1,\varepsilon).$$

THEOREM 1.1. *If ε is sufficiently small,*

$$p^{D_2}(t,x_\varepsilon,y_\varepsilon) > p^{D_1}(t,x_\varepsilon,y_\varepsilon).$$

---

*) The research described here was partially supported by grants from the National Science Foundation.

Proceedings of the International Congress
of Mathematicians, Zürich, Switzerland 1994

It is easy to modify this example to come up with D_1 and D_2 that are bounded, smooth, and strictly convex, and with $x, y \in D_1 \subseteq \bar{D}_1 \subseteq D_2$ such that $p^{D_2}(t,x,y) > p^{D_1}(t,x,y)$.

To prove Theorem 1.1 is not difficult. The method used in [8] obtains an explicit expression for $p^{D_2}(t,0,\cdot)$ from symmetry considerations and then uses translation invariance to get $p^{D_1}(t,x_\varepsilon,y_\varepsilon)$. To estimate $p^{D_2}(t,x_\varepsilon,\cdot)$, we show that it can be written as Uf for a function f, where U denotes the potential with respect to the Green function for reflecting Brownian motion in D_2. Conformal mapping then reduces this to an estimate on reflecting Brownian motion in the upper half plane.

The domains D_1 and D_2 are examples of Lipschitz domains. In two dimensions, conformal mapping is a useful tool, but good estimates for heat kernels with Neumann boundary conditions can be obtained in Lipschitz domains even when the dimension is three or higher, and some upper bounds are available even in Hölder domains [11]. A Hölder domain is one whose boundary is locally the graph of a Hölder continuous function.

2 Boundary Harnack principle

Harnack's inequality says that the values of a positive harmonic function are comparable in the interior of a domain D. Not much can be said, in general, near the boundary. However, if D is a Lipschitz domain, say, and two harmonic functions u and v both vanish on the same portion of the boundary, then u and v tend to 0 near that portion of that boundary at the same rate.

This is called the boundary Harnack principle, and is a very useful tool. For example, it can be used to identify the Martin boundary in Lipschitz domains and to prove Fatou theorems for Lipschitz domains (see [3], [4]). Using probabilistic techniques, it was proved in [2] and [5] that the boundary Harnack principle holds in Hölder domains of order α for $\alpha > 0$. It also holds in twisted Hölder domains of order α provided that $\alpha > 1/2$, where twisted Hölder domains are generalizations of Hölder domains in much the same way that John domains are generalizations of Lipschitz domains.

Let D^c denote the complement of D. As a sample of the kind of theorem that is proved in [2] and [5], we have

THEOREM 2.1. *Suppose D is a Hölder domain of order α, $\alpha > 0$. Let K be a compact set, and V an open set containing K. Let $x_0 \in D$. If u and v are two positive harmonic functions in D such that $u(x_0) = v(x_0) = 1$, u and v vanish at the points of $\partial D \cap V$ that are regular for D^c, and u and v are bounded in a neighborhood of $\partial D \cap V$, then*

$$u(x)/v(x) \leq c, \qquad x \in K \cap D,$$

where c is a constant that depends on D, K, V, and x_0, but not u or v.

If D happens to be the domain above the graph of a Hölder continuous function Γ, and

$$Q = \{(x_1,\ldots,x_d) : \Gamma(x_1,\ldots,x_{d-1}) < x_d < a + \Gamma(x_1,\ldots,x_{d-1}), \\ |(x_1,\ldots,x_{d-1})| \leq R\}$$

for positive numbers a and R, it is not too hard to get estimates for the probability that Brownian motion exits Q through the upper boundary of Q and the probability that Brownian motion exits Q through the sides of Q. The key to the proof of Theorem 2.1 is to combine these estimates to show that Brownian motion conditioned to stay in D is not too likely to exit a box such as Q by creeping along the boundary of D, but rather must go at least a certain amount into the interior of D.

If one replaces Brownian motion by certain diffusions corresponding to divergence form operators, one gets the analogue of Theorem 2.1 with harmonic replaced by L-harmonic, where an L-harmonic function h is one with $Lh = 0$. Here

$$Lh(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial h}{\partial x_j}\Big)(x),$$

and the matrix a_{ij} is assumed to be bounded, uniformly positive definite, and measurable (no smoothness is required). If instead of divergence form operators such as the above, one has nondivergence operators like

$$Lh(x) = \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 h}{\partial x_i \partial x_j}(x)$$

with the same assumptions on a_{ij}, it turns out that the boundary Harnack principle holds for positive L-harmonic functions in Hölder domains of order α when $\alpha > 1/2$. It need not hold for domains when $\alpha < 1/2$, unless the domain satisfies an additional regularity condition. See [9].

3 Martin boundary

Let $x_0 \in D$ and let $g_D(x, y)$ be the Green function for the domain D with pole at y. The (minimal) Martin boundary $\partial_M D$ in a domain is an ideal boundary such that every positive harmonic function h in a domain D can be written

$$h(x) = \int_{\partial_M D} M(x, y)\mu(dy)$$

for some measure μ supported on $\partial_M D$ in one and only one way, where $M(x, y)$, the Martin kernel, is the appropriate extension of $g_D(x, y)/g_D(x_0, y)$. For Lipschitz domains the Martin boundary can be identified with the Euclidean boundary (first proved by Hunt and Wheeden [18]), and it is not hard to show this fact from scaling properties of Lipschitz domains and the boundary Harnack principle. Because Theorem 2.1 says that the boundary Harnack principle holds for Hölder domains, one might wonder whether the Martin boundary must equal the Euclidean boundary in all Hölder domains. The answer is no. However, we have the following theorem from [7].

THEOREM 3.1. *Let*

$$\gamma(x) = bx\frac{\log\log(1/x)}{\log\log\log(1/x)}.$$

Provided b is sufficiently small, the Martin boundary of D may be identified with the Euclidean boundary if D is a C^γ domain.

A C^γ domain is one in which the boundary can be represented locally as the graph of a function whose modulus of continuity is no worse than γ.

This is suggestive of the laws of the iterated logarithm beloved by probabilists, but $x \log\log(1/x)$ is not the right borderline function. In fact for every $b > 0$ there is a C^γ domain with $\gamma(x) = bx \log\log(1/x)$ for which the Martin boundary is different than the Euclidean boundary.

The proof of Theorem 3.1 involves a careful estimate of the constant c in Theorem 2.1 for C^γ domains.

4 Conditional lifetimes

Around 1983 K.L. Chung posed the conditional lifetime problem. Suppose $\mathbb{E}_z^x$ represents the expectation of Brownian motion in a domain D conditioned to exit D at the point z. Let τ_D represent the exit time of D for Brownian motion. The conditional lifetime problem is to determine for which domains $\mathbb{E}_z^x \tau_D$ is finite.

Cranston and McConnell [16] showed that in two dimensions, the expected conditional lifetime is bounded by a constant that depends only on the area of D and not on x or z. In three dimensions the lifetime can be infinite even for bounded domains [16], but must be finite in bounded Lipschitz domains [14]. Various refinements have been obtained, and in the last few years, some extensive generalizations have been proved ([1], [6], [17]). For example, we have

THEOREM 4.1. *Suppose $p > d - 1$ and D is a domain of the form*

$$D = \{(x_1, \ldots, x_d) : |(x_1, \ldots, x_{d-1})| < 1,\ 0 > x_d > -f(x_1, \ldots, x_{d-1})\},$$

where f is a nonnegative function that is in $L^p(\mathbb{R}^{d-1})$. Then $\mathbb{E}_z^x \tau_D < c$, where c is a finite constant depending only on f and not x or z.

Many refinements of Theorem 4.1 are possible. Brownian motion can be replaced by diffusions corresponding to operators L in either divergence or nondivergence form. Twisted Hölder domains of order α are possible (it turns out that the critical α becomes 1/3). A domain need only have its boundary be given locally by the graph of an L^p function.

Various proofs of theorems such as Theorem 4.1 have been given. In [17], the Girsanov transformation is the principal tool; in [1] heat kernel estimates play an important part. The proof in [6] is based on the observation that the amount of time Brownian motion spends in a strip of width r is proportional to r^2. Combining this estimate with the techniques used in [14] and [16] give Theorem 4.1 and its relatives.

5 Conditional gauge

In solving the Dirichlet problem using probability theory, it is possible to replace $(1/2)\Delta u(x)$ by $(1/2)\Delta u(x) + q(x)u(x)$, the Schrödinger operator, if one uses the Feynman-Kac formula. Expectations of expressions such as $\exp(\int_0^{\tau_D} q(X_s)\, ds)$ then play a role. To study local behavior of the solution to the Dirichlet problem, such as the Fatou theorems, Martin boundary, harmonic measure, etc., one uses conditioned Brownian motion. It should be no surprise, then, that to study the local behavior of solutions to equations involving the Schrödinger operator, it is necessary to look at $\mathbb{E}_z^x \exp(\int_0^{\tau_D} q(X_s)\,ds)$. Here X_s is Brownian motion, τ_D is the exit time from D, and $\mathbb{E}_z^x$ represents the expectation of Brownian motion started at x and conditioned to exit D at z.

The above expectation is called the conditional gauge. It need not be finite even when q is constant and D is an interval in one dimension. However, when it is finite, many of the local properties of harmonic functions carry over to functions that are harmonic with respect to the operator $(1/2)\Delta + q$.

The conditional gauge theorems are results that say that under certain assumptions on q and D, if the conditional gauge is finite for one pair (x, z) with $x \neq z$, then it is finite for all pairs (x, z), and the values of the conditional gauge are comparable. For example, if $d \geq 3$, D is a Lipschitz domain in $\mathbb{R}^d$, and q is in the Kato class, then the conditional gauge theorem holds [15]. For q to be in the Kato class means that

$$\limsup_{\varepsilon \to 0} \sup_{x \in D} \int_{D \cap B(x,\varepsilon)} |q(y)|\,|x-y|^{2-d}\,dy = 0,$$

where $B(x, \varepsilon)$ is the ball of radius ε about x. When $d = 2$, $|x - y|^{2-d}$ in the definition is replaced by $\log(1/|x - y|)$.

Various partial results have been proved for the case $d = 2$, but by analogy to the conditional lifetime results, one would expect that the conditional gauge theorem ought to be true in bounded domains in the plane, with no further assumptions on the domain necessary. This turns out to be correct [10].

The key to the conditional gauge theorem is to get a sufficiently good estimate on the Green function for conditioned Brownian motion in D, namely on the ratio

$$\frac{g_D(x,y)g_D(y,z)}{g_D(x,z)},$$

where g_D is the Green function for ordinary Brownian motion in D. In [10] we obtain the bound

THEOREM 5.1. *If D is a bounded domain in the plane, then*

$$\frac{g_D(x,y)g_D(y,z)}{g_D(x,z)} \leq c\big[1 + \log^+(1/|x-y|) + \log^+(1/|y-z|)\big],$$

where c depends only on the diameter of the domain and $\log^+ w = \max(0, \log w)$.

Actually, certain unbounded domains are also allowed, such as domains contained in a strip or domains that have finite area.

If x is a distance 1 from a point y, then the probability that Brownian motion starting at x makes a loop around y before exiting $B(y, 2)$ or hitting $B(y, 1/2)$ is comparable to the probability of hitting $B(y, 1/2)$ before exiting $B(y, 2)$. The key to Theorem 5.1 is to show that the same is true even if in addition we kill Brownian motion on hitting a set K, provided the capacity of K is sufficiently small.

References

[1] R. Bañuelos, *Intrinsic ultracontractivity and eigenvalue estimates for Schrödinger operators*, J. Funct. Anal. **100** (1991), 181–206.

[2] R. Bañuelos, R. F. Bass, and K. Burdzy, *Hölder domains and the boundary Harnack principle*, Duke Math. J. **64** (1991), 195–200.

[3] R. F. Bass, Probabilistic Techniques in Analysis, Springer-Verlag, Heidelberg, 1995.

[4] R. F. Bass and K. Burdzy, *A probabilistic proof of the boundary Harnack principle*, Seminar on Stochastic Processes 1989, 1–16, Birkhäuser, Boston, 1990.

[5] ——— *A boundary Harnack principle for twisted Hölder domains*, Ann. of Math. (2) **134** (1991), 253–276.

[6] ——— *Lifetimes of conditioned diffusions*, Probab. Theory Related Fields **91** (1992), 405–444.

[7] ——— *The Martin boundary in non-Lipschitz domains*, Trans. Amer. Math. Soc. **337** (1993), 361–378.

[8] ——— *On domain monotonicity of the Neumann heat kernel*, J. Funct. Anal. **116** (1993), 215–224.

[9] ——— *The boundary Harnack principle for non-divergence form operators*, J. London Math. Soc. (2), **50** (1994), 157–169.

[10] ——— *Conditioned Brownian motion in planar domains*, Probab. Theory Related Fields **101** (1995), 479–493.

[11] R. F. Bass and P. Hsu, *Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains*, Ann. Probab. **19** (1991), 486–508.

[12] R. A. Carmona and W. Zheng, *Reflecting Brownian motions and comparison theorems for Neumann heat kernels*, J. Funct. Anal. **123** (1994), 109–128.

[13] I. Chavel, *Heat diffusion in insulated convex domains*, J. London Math. Soc. (2) **34** (1986), 473–478.

[14] M. Cranston, *Lifetime of conditioned Brownian motion in Lipschitz domains*, Z. Wahrsch. **70** (1985), 335–340.

[15] M. Cranston, E. Fabes, and Z. Zhao, *Conditional gauge and potential theory for the Schrödinger operator*, Trans. Amer. Math. Soc. **307** (1988), 171–194.

[16] M. Cranston and T. R. McConnell, *The lifetime of conditioned Brownian motion*, Z. Wahrsch. **65** (1983), 1–11.

[17] B. Davis, *Intrinsic ultracontractivity and the Dirichlet Laplacian*, J. Funct. Anal. **100** (1991), 163–180.

[18] R. A. Hunt and R. L. Wheeden, *Positive harmonic functions on Lipschitz domains*, Trans. Amer. Math. Soc. **132** (1970), 507–527.

[19] W. S. Kendall, *Coupled Brownian motions and partial domain monotonicity for the Neumann heat kernel*, J. Funct. Anal. **86** (1989), 226–236.