On Aronson's upper bounds for heat kernels

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Abstract. Let $\mathcal{L}$ be a uniformly elliptic operator in divergence form on $\mathbb{R}^d$ and let $p(t,x,y)$ be the fundamental solution to the heat equation for $\mathcal{L}$. A new proof is given of Aronson's upper bound: $p(t,x,y) \le c_1 t^{-d/2} \exp(-c_2|x-y|^2/t)$.

Let $\mathcal{L}$ be the elliptic operator on $\mathbb{R}^d$ in divergence form given by

$$\mathcal{L}f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial f}{\partial x_j}\Big)(x).$$

We assume the operator is uniformly elliptic: there exists $\Lambda > 0$ such that

$$\Lambda \sum_{i=1}^d y_i^2 \le \sum_{i,j=1}^d a_{ij}(x) y_i y_j \le \Lambda^{-1} \sum_{i=1}^d y_i^2, \qquad x \in \mathbb{R}^d, \quad (y_1,\ldots,y_d) \in \mathbb{R}^d. \tag{1}$$

One of the most important facts concerning such operators is Aronson's bounds: if $p(t,x,y)$ is the fundamental solution to the heat equation $\partial u/\partial t = \mathcal{L}u$ in $\mathbb{R}^d$, then there exist constants c_1, c_2, c_3, c_4 such that

$$p(t,x,y) \le c_1 t^{-d/2} \exp(-c_2|x-y|^2/t), \qquad x,y \in \mathbb{R}^d, \quad t>0, \tag{2}$$

and

$$c_3 t^{-d/2} \exp(-c_4|x-y|^2/t) \le p(t,x,y), \qquad x,y \in \mathbb{R}^d, \quad t>0. \tag{3}$$

Aronson's proof [**1**] relies on Moser's Harnack inequality. Another proof was given by Fabes-Stroock [**5**] based on a method developed by Davies. A global upper bound on $p(t,x,y)$ of the form

$$p(t,x,y) \le c_1 t^{-d/2}, \qquad x,y \in \mathbb{R}^d, \quad t>0, \tag{4}$$

is not hard; the proof can be found in [**4**, Sect. VII.4], [**5**], or [**7**]. Once one has the upper bound (2), the lower bound (3) can then be established; see [**5**]. The bounds (2) and (3) together imply Moser's parabolic Harnack inequality; again, see [**5**].

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By far the hardest part of Aronson's bounds is proving (2) when $|x-y|^2>t$. The purpose of this paper is to give a simpler proof of this part of the upper bound. We base our proof on an idea in Nash' paper [**7**] (a different idea, however, than the one referred to in the title of [**5**]), together with simplifications of some techniques used in [**2**] and [**3**].

We will assume the a_{ij} satisfy (1). We will also assume for convenience that they are smooth, but none of our estimates will depend on the smoothness; the case of non-smooth a_{ij} will follow from our result by an easy limit argument. We will also assume for simplicity that the coefficients of $\mathcal{L}$ do not depend on t; only cosmetic changes are needed to include this case in the proof.

If X_t is the diffusion associated to the elliptic operator $\mathcal{L}$ with coefficients $a_{ij}(x)$ (see [**4**, Chap. VII]), then rX_{t/r^2} will be the diffusion associated to the elliptic operator with coefficients $a_{ij}(rx)$, which satisfy the same boundedness and ellipticity bounds as $a_{ij}(x)$; we refer to this property as scaling.

Under the assumption that the a_{ij} are smooth, the diffusion X_t associated with $\mathcal{L}$ can be written

$$X_t = X_0 + \int_0^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds,$$

where W_t is a d-dimensional Brownian motion, the first integral on the right is an Itô stochastic integral, σ is a bounded positive definite square root of a, and the ith component of b is $\sum_{j=1}^d \partial a_{ji}/\partial x_j$; see [**4**, Section I.2]. It is easy to deduce from this (cf. the proof of Theorem I.3.5 in [**4**]) that for each x_0 and each t

$$\mathbb{E}^{\,x_0} \sup_{s\le t} |X_s - x_0|^2 < \infty. \tag{5}$$

The letter c with subscripts will denotes finite positive constants whose exact values are unimportant.

Our first theorem is the second main result of [**7**]. Let $x_0\in\mathbb{R}^d$,

$$M(t) = \int |y-x_0| p(t,x_0,y)dy, \qquad Q(t) = -\int p(t,x_0,y)\log p(t,x_0,y)dy.$$

Since $M(t)=\mathbb{E}^{\,x_0}|X_t-x_0|$, then $M(t)$ is finite by (5). The finiteness of $Q(t)$ will follow from (6) and (7) below.

Theorem 1. *There exists c_1 not depending on x_0 or t such that $M(t)\le c_1 t^{1/2}$.*

Proof. First, using (4) and the fact that $\int p(t,x_0,y)dy=1$, we have

$$Q(t) \ge -c_2 + \tfrac{1}{2} d \log t. \tag{6}$$

Second, note $\inf_s(s\log s + \lambda s) = -e^{-\lambda-1}$. Using this with $\lambda = a|y-x_0|+b$, we obtain

$$-Q(t) + aM(t) + b = \int [p(t,x_0,y)\log p(t,x_0,y) + (a|y-x_0|+b)p(t,x_0,y)]dy$$
$$\geq -e^{-b-1}\int e^{-a|y-x_0|}dy = -c_3 e^{-b}a^{-d}.$$

Setting $a = d/M(t)$ and $e^{-b} = (e/c_3)a^d$, after some algebra we obtain

$$M(t) \geq c_4 e^{Q(t)/d}. \tag{7}$$

Third, we differentiate $Q(t)$. Since the a_{ij} are smooth and (1) holds, it is well known ([**6**]) that $p(t,x_0,y)$ is strictly positive and is C^∞ in each variable on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and that $p(t,x_0,y)$ and its first and second partial derivatives have exponential decay at infinity. Performing the differentiation,

$$Q'(t) = -\int(1+\log p(t,x_0,y))\frac{\partial}{\partial t}p(t,x_0,y)dy$$
$$= -\int(1+\log p(t,x_0,y))\mathcal{L}p(t,x_0,y)dy.$$

Applying the Gauss-Green formula on the ball of radius K about 0, letting $K\to\infty$, and using the exponential decay of $p(t,x_0,y)$ and its first and second partial derivatives, the above expression is equal to

$$\int \nabla(\log p(t,x_0,y))\cdot a\nabla p(t,x_0,y)dy$$
$$= \int[\nabla(\log p(t,x_0,y))\cdot a\nabla(\log p(t,x_0,y))]p(t,x_0,y)dy.$$

Using Cauchy-Schwarz and the uniform ellipticity bounds,

$$Q'(t) \geq c_5\Big(\int|\nabla\log p(t,x_0,y)|p(t,x_0,y)dy\Big)^2 = c_5\Big(\int|\nabla p(t,x_0,y)|dy\Big)^2.$$

Set $r(y) = |y-x_0|$. As $|\nabla r|\leq 1$, we have similarly that $M'(t) = -\int\nabla r\cdot a\nabla p(t,x_0,y)dy$, and so

$$|M'(t)| \leq \int|a\cdot\nabla p(t,x_0,y)|dy.$$

(Because r is not differentiable at x_0, to establish this we approximate r by smooth functions and use a simple limit argument.) We thus conclude

$$Q'(t) \geq c_6(M'(t))^2. \tag{8}$$

By the continuity of X_t, (5), and dominated convergence,

$$\lim_{t\to 0} M(t) = \lim_{t\to 0} \mathbb{E}^{x_0}|X_t - x_0| = 0,$$

so

$$c_4 e^{Q(t)/d} \le M(t) \le c_7 \int_0^t (Q'(s))^{1/2} ds. \tag{9}$$

Finally, define $R(t) = d^{-1}[Q(t) + c_2 - \frac{d}{2}\log t]$, and observe from (6) that $R(t) \ge 0$. Then

$$Q'(t) = dR'(t) + d/(2t).$$

Using (9) and the inequality $(a+b)^{1/2} \le a^{1/2} + b/(2a^{1/2})$, we have

$$\begin{aligned} c_8 t^{1/2} e^{R(t)} \le M(t) &\le c_9 d^{1/2} \int_0^t \Big(\frac{1}{2s} + R'(s)\Big)^{1/2} ds \\ &\le c_{10} \int_0^t \Big(\frac{1}{2s}\Big)^{1/2} ds + c_{10} \int_0^t \Big(\frac{s}{2}\Big)^{1/2} R'(s) ds. \end{aligned}$$

By integration by parts and the fact that $R \ge 0$, this is less than

$$c_{11}(2t)^{1/2} + c_{11} R(t) \Big(\frac{t}{2}\Big)^{1/2},$$

which leads to

$$c_8 e^{R(t)} \le M(t)/t^{1/2} \le c_{12}(1 + R(t)).$$

The inequality $c_8 e^{R(t)} \le c_{12}(1 + R(t))$ implies that $R(t)$ is bounded, and the result follows. □

Theorem 2. *There exist c_1 and c_2 such that (2) holds for all $x, y \in \mathbb{R}^d$, $t > 0$.*

Proof. First, if $S_\lambda = \inf\{t : |X_t - X_0| \ge \lambda\}$, then

$$\begin{aligned} \mathbb{P}^x(\sup_{s\le t} |X_s - X_0| \ge \lambda) &\le \mathbb{P}^x(S_\lambda \le t, |X_t - X_0| \ge \lambda/2) + \mathbb{P}^x(S_\lambda \le t, |X_t - X_0| \le \lambda/2) \\ &\le \mathbb{P}^x(|X_t - X_0| \ge \lambda/2) + \int_0^t \mathbb{P}^x(|X_t - X_s| \ge \lambda/2, S_\lambda \in ds). \end{aligned}$$

By Chebyshev's inequality and Theorem 1, the first term on the right hand side is bounded by

$$\frac{2\mathbb{E}^x|X_t - X_0|}{\lambda} \le \frac{2M(t)}{\lambda} \le \frac{c_3 t^{1/2}}{\lambda}.$$

By the strong Markov property, the second term is bounded by

$$\int_0^t \mathbb{E}^x[\mathbb{P}^{X_s}(|X_{t-s} - X_0| \ge \lambda/2); S_\lambda \in ds] \le \frac{2}{\lambda} \int_0^t M(t-s) \mathbb{P}^x(S_\lambda \in ds) \le \frac{c_3 t^{1/2}}{\lambda}.$$

Adding,

$$\mathbb{P}^x(\sup_{s\le t}|X_s - X_0| \ge \lambda) \le \frac{2c_3 t^{1/2}}{\lambda}. \tag{10}$$

Second, let $D > 0$, let $n = [aD^2]$, and let $b > 0$, where a, b will be chosen in a moment. By (10) we have

$$\begin{aligned}
\mathbb{E}^x e^{-nS_{D/n}} &\le 1 \cdot \mathbb{P}^x(S_{D/n} < b/n) + e^{-n(b/n)}\mathbb{P}^x(S_{D/n} > b/n)\\
&= (1 - e^{-b})\mathbb{P}^x(S_{D/n} < b/n) + e^{-b}\\
&\le (1 - e^{-b})\frac{2c_3(b/n)^{1/2}}{D/n} + e^{-b}\\
&\le e^{-2}
\end{aligned}$$

if we first choose b large and then a small (a, b can be chosen independently of x, D and n). Let $T_0 = 0$ and define inductively $T_{i+1} = T_i + S_{D/n} \circ \theta_{T_i}$, where the θ_t are the usual shift operators for Markov processes; this means that T_i is the ith time that X_t moves a distance D/n. By the strong Markov property

$$\mathbb{E}^x e^{-nT_m} = \mathbb{E}^x[e^{-nT_{m-1}}; \mathbb{E}^{X_{T_{m-1}}} e^{-nS_{D/n}}] \le e^{-2}\mathbb{E}^x e^{-nT_{m-1}},$$

so by induction

$$\mathbb{E}^x e^{-nT_n} \le e^{-2n}.$$

Then

$$\begin{aligned}
\mathbb{P}^x(\sup_{s\le 1}|X_s - X_0| \ge D) &\le \mathbb{P}^x(T_n \le 1) = \mathbb{P}^x(e^{-nT_n} \ge e^{-n}) \\
&\le e^n\mathbb{E}^x e^{-nT_n} \le e^n e^{-2n} = e^{-n} \le e^{-c_4 D^2}.
\end{aligned} \tag{11}$$

Third, let $A = \{z : |z - x| > |z - y|\}$. By (11) with $D = |x - y|/2$ and (4)

$$\begin{aligned}
\int_A p(1, x, z)p(1, z, y)dz &\le c_5 \int_A p(1, x, z)dz = c_5\mathbb{P}^x(X_1 \in A)\\
&\le c_5\mathbb{P}^x(|X_1 - x| \ge |x - y|/2) \le c_5 e^{-c_6|x-y|^2},
\end{aligned}$$

while

$$\int_{A^c} p(1, x, z)p(1, z, y)dz \le c_5\mathbb{P}^y(X_1 \in A^c) \le c_5\mathbb{P}^y(|X_1 - y| \ge |x - y|/2) \le c_5 e^{-c_6|x-y|^2}.$$

Adding and using the semigroup property,

$$p(2, x, y) = \int p(1, x, z)p(1, z, y)dz \le 2c_5 e^{-c_6|x-y|^2}.$$

The theorem now follows by scaling. □

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