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Probability Theory and Related Fields

Uniqueness for Diffusions with Piecewise Constant Coefficients

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Summary. Let L be a second-order partial differential operator in $\mathbf{R}^d$. Let $\mathbf{R}^d$ be the finite union of disjoint polyhedra. Suppose that the diffusion matrix is everywhere non singular and constant on each polyhedron, and that the drift coefficient is bounded and measurable. We show that the martingale problem associated with L is well-posed.

1. Introduction

Let L be the operator defined on $f \in C^2(\mathbb{R}^d)$ by

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial f(x)}{\partial x_i},$$

where the a_{ij} and b_i are bounded and measurable and where the matrix a is uniformly positive definite. When the a_{ij} are continuous, Stroock and Varadhan (see [6]) showed that there is one and only one continuous strong Markov process corresponding to L by showing that there is a unique solution to the martingale problem:

(1.1) for each x_0 there is exactly one probability measure P on $C([0, \infty), \mathbb{R}^d)$ such that

$$P(X_0 = x_0) = 1$$

and

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds \quad \text{is a } P \text{ local martingale for all } f \in C^2(\mathbb{R}^d).$$

When a is discontinuous, existence is known to hold regardless of the dimension d ([6], Exercise 12.4.3), while uniqueness holds when d is 1 and 2 ([6],

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Exercises 7.3.3 and 7.3.4). For $d \geqq 3$ the argument of Stroock and Varadhan implies uniqueness when

$$\sup_x \sup_{i,j \leqq d} |a_{ij}(x) - \delta_{ij}| < \varepsilon_d,$$

where ε_d is a fixed number depending on the dimension. This last result can be extended slightly by using a localization argument, but the question of uniqueness for general discontinuous a remains open.

In this paper, we prove uniqueness for an interesting special case. We consider the case where $\mathbb{R}^d$ can be divided up into finitely many polyhedra and where $a(x)$ is constant in the interior of each polyhedron. We make no restrictions on the eigenvalues of $a(x)$, other than that they be positive, nor on the number of polyhedra. Our motivation comes from the so called "piece-wise linear" filtering problem, see [5].

Even in this special case, some interesting phenomena occur. For example it is possible for such a process to hit points (see Sect. 3). Indeed that possibility causes the main difficulty in our proof of uniqueness (see Sect. 5).

The question of uniqueness in the neighborhood of a vertex point is related to a problem of Varadhan and Williams [7] concerning Brownian motion in a wedge. We could not use the technique of [7] since explicit formulae for the Green's function are not available. Instead we make use of the Krein-Rutman theorem for positive operators and an ergodic theory argument. We expect that our method could be modified to give a new proof of [7], and could perhaps lead to higher dimensional analogs.

After our research was completed, we learned of the article [8] by Williams. She considers Brownian motion with polar drift; this problem, although quite different than ours, has many similarities to the situation of Sect. 5. Our techniques, however, are different. We use the Krein-Rutman theorem to give a ratio limit theorem for nonconservative Markov chains. As far as we know, this is new and is of independent interest.

In Sect. 2 we give some preliminaries and state our main result, Theorem 2.1. In Sect. 3 we present an example of a diffusion that hits 0 infinitely often, a.s. We classify the boundary points of the polyhedra as being nonvertex or vertex, and we show uniqueness for the martingale problem in neighborhoods of such points in Sects. 4 and 5, respectively. Finally, in Sect. 6 we complete the proof of Theorem 2.1.

Note that Theorem 5.5, which proves uniqueness whenever $a(x) = a(x/|x|)$ and there is uniqueness up to the first time that the process hits zero, covers other cases than the one considered in the other sections of the paper.

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2. Preliminaries

Suppose $a: \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is measurable, uniformly bounded, and uniformly positive definite: there exists $\lambda_0 > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\, y_i\, y_j \geqq \lambda_0 \sum_{i=1}^{d} y_i^2$$

for all $y_1, \ldots, y_d$. Suppose $b\colon \mathbb{R}^d \to \mathbb{R}^d$ is measurable, locally bounded, and grows at most like $c|x|$ at infinity. Let

$$L_{a,b} f(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f}{\partial x_i\, \partial x_j}(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x_i}(x). \tag{2.1}$$

Let $\Omega = C([0, \infty], \mathbb{R}^d)$, and let $X_t(\omega) = \omega(t)$. We say a probability measure P satisfies the martingale problem for $L_{a,b}$ starting at $x_0 \in \mathbb{R}^d$ if

$$P(X_0 = x_0) = 1 \tag{2.2}$$

and

$$f(X_t) - f(X_0) - \int_0^t L_{a,b} f(X_s)\, ds \tag{2.3}$$

is a P-local martingale for all $f \in C^2(\mathbb{R}^d)$.

When b is identically 0, we will just refer to the martingale problem for L_a or even the martingale problem for a. Saying P is uniquely determined means that any two solutions to the martingale problem for $L_{a,b}$ agree on $F_\infty = \sigma(X_t, t \in [0, \infty))$.

We suppose $\mathbb{R}^d$ can be divided up into finitely many polyhedra $A_1, \ldots, A_n$ such that a is constant in the interior of each A_i. So $\mathbb{R}^d = \bigcup_{1 \leqq i \leqq n} \bar{A}_i$ and the A_i have pairwise disjoint interiors. Since the d-dimensional Lebesgue measure of $\bigcup_{1 \leqq i \leqq n} \partial A_i$ is 0 and a is assumed uniformly positive definite on $\mathbb{R}^d$, we have that the process X_t spends 0 time in the boundaries of the A_i [9] and hence it follows that the value of a on the boundaries is immaterial (still assuming nondegeneracy). For convenience, we take $a(x) = 1$, the identity matrix, for $x \in \bigcup_{1 \leqq i \leqq n} \partial A_i$.

We need to distinguish two types of boundary points. Let us say that $x \in \bigcap_{i=1}^{n} \partial A_i$ is a nonvertex boundary point if there exists an integer $k < d$ and a coordinate system for a neighborhood of x such that $a(y)$ depends only on the first k coordinates of y for y in the neighborhood. Otherwise we call x a vertex boundary point.

Our main result, proven in Sect. 6, is:

Theorem 2.1. *Suppose a is measurable, uniformly bounded, and uniformly positive definite, b measurable and locally bounded with at most linear growth in $|x|$ at infinity, $x_0 \in \mathbb{R}^d$. Suppose $\mathbb{R}^d$ can be divided up into finitely many polyhedra so that a is constant in the interior of each polyhedron. Then there exists a solution to the martingale problem for $L_{a,b}$ starting at x_0 and that solution is unique.*

We will use the notation

$$\tau(r)=\inf\{t: |X_t|=r\}, \qquad \tau(0)=\inf\{t: |X_t|=0\}.$$

The transpose of a matrix A will be denoted A^*, the inner product in $\mathbb{R}^d$ by $\langle \ , \ \rangle$. As is customary, a positive definite matrix must be symmetric. The letter c will denote constants whose values are unimportant and may change from place to place.

3. An Example

In this section we want to construct an example of a solution to a martingale problem that hits 0 infinitely often with probability 1. First recall that if $\tilde{W}$ is a one dimensional Brownian motion and Z_t is the solution to

$$dZ_t=d\tilde{W}_t+(2Z_t)^{-1}(v-1)\,dt, \qquad Z_0 \neq 0, \qquad \text{for } t<\tau(0),\ Z_t=0 \text{ for } t\geqq\tau(0), \tag{3.1}$$

so that Z is a Bessel process of order v, then Z_t will hit 0 in finite time if $v<2$. Perhaps the easiest way of seeing this is to realize by Ito's formula that Z_t^{2-v} is a martingale, degenerate only when $Z_t=0$. So it is a time change of a Brownian motion, and therefore.

$$P(Z_t \text{ hits } \varepsilon \text{ before } M\,|\,Z_0=x_0)=\frac{M^{2-v}-x_0^{2-v}}{M^{2-v}-\varepsilon^{2-v}}, \ \varepsilon<x_0<M.$$

Now let $\varepsilon\to 0$, then $M\to\infty$.

We divide $\mathbb{R}^d$ into finitely many polyhedra $A_1, \ldots, A_n$ such that each polyhedron is a cone with vertex 0: if $x\in A_i$, then $rx\in A_i$, for all $r>0$. We choose the polyhedra in such a way that there exists a number $\varepsilon\in[0, 1/2]$ (to be chosen later) and points $x_i\in A_i$ such that $|x_i|=1$ and for all $x\in A_i$, $|x|^{-2}\langle x, x_i\rangle^2\geqq 1-\varepsilon$. This condition implies that the aperture of each cone is small.

Define σ on the interior of A_i to be the positive definite matrix whose largest eigenvalues is of size 1 with corresponding eigenvector x_i and the remaining $d-1$ eigenvalues of size ε. Define $\sigma(x)$ for $x\in\bigcup_j \partial A_j$ to be equal to the value of σ on $A_{j(x)}$, where $j(x)$ is the smallest index j such that $x\in\partial A_j$. Our definition of σ implies that trace $\sigma\sigma^*=1+(d-1)\,\varepsilon^2$, while given δ, we can choose ε sufficiently small so that

$$\big||x|^{-2}x^*\sigma\sigma^*x-1\big|\leqq\delta \qquad \text{for } x\neq 0. \tag{3.2}$$

Let $a=\sigma\sigma^*$ and let P be a solution to the martingale problem for a starting from $x_0\neq 0$. If $W_t=\int_0^t \sigma^{-1}(X_s)\,dX_s$ so that $dX_t=\sigma(X_t)\,dW_t$, then under P, W_t is a Brownian motion in $\mathbb{R}^d$. We will show $P(\tau(0)<\infty)=1$. Since $x_0\neq 0$ is arbitrary, an elementary renewal argument then implies that $P(X_t$ hits 0 i.o.$)=1$ for $x_0\neq 0$.

Let $R_t=|X_t|$. By Ito's formula, for $t<\tau(0)$ we have

$$dR_t=|X_t|^{-1}X_t^*\sigma(X_t)\,dW_t+(2R_t)^{-1}(\operatorname{trace}\sigma\sigma^*(X_t)\\ -|X_t|^{-2}X_t^*\sigma\sigma^*(X_t)X_t)\,dt. \tag{3.3}$$

Let $A(t)=\int_0^t|X_s|^{-2}X_s^*\sigma\sigma^*(X_s)X_s\,ds$ and let $S_t=R_{A^{-1}(t)}$. By (3.2), $dA_t/dt\in[1-\delta, 1+\delta]$, and so clearly $P[\tau(0)<\infty]=1$ if and only if $P(S_t$ hits 0 in finite time$)=1$.

Now S_t is a semimartingale with $\langle S,S\rangle_t=t$. Hence the martingale part of S is a Brownian motion, say $\tilde{W}$. Moreover from (3.3) we see that

$$dS_t=d\tilde{W}_t+\frac{C_t}{2S_t}\,dt,$$

where $C_t\leqq[(d-1)\varepsilon^2+\delta](1-\delta)^{-1}$, a.s.

Choose $\delta<1$, then ε, both sufficiently small so that $\nu=1+[(d-1)\varepsilon^2+\delta](1-\delta)^{-1}$ is strictly less than 2. Let Z_t be the solution to (3.1) satisfying $Z_0=|x_0|$. By a comparison theorem for stochastic differential equations ([2], Theorem VI.1.1) together with a localization argument, $S_t\leqq Z_t$, a.s. for $t\leqq\rho_n=\inf\{t\colon Z_t\leqq 1/n$ or $S_t\leqq 1/n\}$. Since n is arbitrary and Z_t hits 0 with probability 1, S_t must also hit 0 with probability 1.

4. Nonvertex Boundary Points

We begin with a proposition that may seen obvious. The difficulty is that if Y_i is a weak solution to $dY_i=\sigma_i(Y_i)\,dW_i$, $i=1,2$, where W_1 and W_2 are independent Brownian motions, then it is possible that the σ-field generated by Y_i is strictly larger than the σ-field generated by W_i and so the independence of W_1 and W_2 does not immediately imply that of Y_1 and Y_2.

Proposition 4.1. *Suppose the solution to the martingale problem for the $k\times k$ matrix a starting from y_0 is unique. Then the solution to the martingale problem for $\tilde{a}$ starting from (y_0,z_0) is unique, where $\tilde{a}=\begin{pmatrix}a&0\\0&I\end{pmatrix}$, I is the $(d-k)\times(d-k)$ identity matrix, and z_0 is any point in $\mathbb{R}^{d-k}$.*

Proof. Let Y be the first k coordinates of X. Let σ be the positive definite square root of a and let $\tilde{\sigma}=\begin{pmatrix}\sigma&0\\0&I\end{pmatrix}$. Let P be any solution to the martingale problem for $\tilde{a}$ starting at $x_0=(y_0,z_0)$. Let $W_t=\int_0^t\tilde{\sigma}^{-1}(X_s)\,dX_s$, and write $W=\begin{pmatrix}W'\\W''\end{pmatrix}$, where W' is the first k coordinates, W'' the remaining $d-k$ coordinates. Note $X=(Y,W'')$.

Then under P, W is a d-dimensional Brownian motion. Let P_a be the unique solution to the martingale problem for a starting at y_0.

Now let $F_t=\sigma(Y_s, s\in[0,t])$, $G=\sigma(W_s'', s\in[0,\infty))$. Let Q_ω be a regular conditional probability for $P(\ |G)$; that is, for any $A\in\sigma(X_s, s\in[0,\infty))$, we have $P(A|G)=Q_\omega(A)$, a.s. (dP).

Suppose $A\in F_s$, $B\in\sigma(W_r'', r\in[0,s])$, $C\in\sigma(W_r''-W_s'', r\geqq s)$. As is well-known (see, for example, [10, III.10]), W_t-W_s is independent of $\sigma(Y_r, W_r''; r\in[0,s])$. Using this and the independence of W' and W'', we have

$$E(W_t'-W_s'; A\cap B\cap C)=E(W_t'-W_s'; C)\,P(A\cap B)=0.$$

By linearity and a limit argument, we see that whenever $B\in G$, $A\in F_s$, and H_s is G_s-predictable and bounded, where $G_t=\sigma(F_t, G)$, then

$$\int_B Q_\omega\left(\int_s^t H_r\,dW_r'; A\right)P(d\omega)=E\left(E\left(\left(\int_s^t H_r\,dW_r'; A\right)\Big/ G\right); B\right)$$
$$=0.$$

A routine argument using the continuity of the stochastic integral to handle the null sets shows that for almost all $\omega(dP)$, $\int_0^t H_s\,dW_s'$ is a Q_ω-martingale.

Now let $f\in C^2(\mathbb{R}^k)$. By Ito's formula,

$$f(Y_t)-f(Y_0)-\int_0^t L_a f(Y_s)\,ds=\int_0^t \nabla f(Y_s)\,dY_s \tag{4.1}$$
$$=\int_0^t \nabla f(Y_s)\,\sigma(Y_s)\,dW_s'.$$

The right side of (4.1) is a Q_ω local martingale, a.s. (dP) by the above. Since $C^2(\mathbb{R}^k)$ is separable, it follows that for almost all $\omega(dP)$, Q_ω is a solution to the martingale problem for a, hence $Q_\omega=P_a$, a.s.

Since the law of W'' is uniquely characterized as that of a $(d-k)$ dimensional Brownian motion, and the conditional law of Y under P, given $\sigma(W'')$, is P_a, then the joint law of (Y, W'') under P is characterized uniquely. □

Since Q_ω is deterministic, the above proof shows that Y and W'' are in fact independent.

We now want to give a decomposition of positive definite matrices.

Lemma 4.2. *Suppose a is a $d\times d$ positive definite matrix, $k<d$. Then there exists a $d\times d$ matrix σ of the form $\sigma=\begin{pmatrix}A & O\\ B & C\end{pmatrix}$, where A and C are positive definite, A is $k\times k$, C is $(d-k)\times(d-k)$, and $\sigma\sigma^*=a$.*

Proof. Write $a=\begin{pmatrix}D & F^*\\ F & G\end{pmatrix}$, where D is $k\times k$. Let A be the positive definite square root of D and let $B=FA^{-1}$. The matrix $G-BB^*$ is symmetric. Provided we show $G-BB^*$ is positive definite, letting C be the positive definite square root of $G-BB^*$ will complete the proof.

Let Z be $(d-k)\times 1$, let $Y=-A^{-1}B^*Z$, and let $X=\begin{pmatrix}Y\\Z\end{pmatrix}$. Since a is positive definite, there exists a constant λ_0 such that $X^*aX\geqq\lambda_0 X^*X$. But $XX^*=YY^*+ZZ^*\geqq Z^*Z$. And direct calculation using the definitions of A, B, and Y shows that

$$X^*aX=Z^*(G-BB^*)Z. \quad \square$$

We now come to the main result of this section.

Theorem 4.3. *Let $k<d$ and suppose $a(x)$ depends only on $x_1,\dots,x_k$, the first k coordinates of x. Suppose $a=\begin{pmatrix}D & F^*\\ F & G\end{pmatrix}$, where D is $k\times k$, and suppose the solution to the martingale problem starting from y_0 for D is unique. Then the solution to the martingale problem for a starting from (y_0, z_0) is unique for all $z_0\in\mathbb{R}^{d-k}$.*

Proof. Choose σ as in Lemma 4.2. Since a is nondegenerate, σ is invertible. Let P be a solution to the martingale problem for a starting from (y_0, z_0). Let $W_t=\int_0^t\sigma^{-1}(X_s)\,dX_s$. Write $W=\begin{pmatrix}W'\\W''\end{pmatrix}$, $X=\begin{pmatrix}Y\\Z\end{pmatrix}$, where W' and Y are $k\times 1$. Then under P, W is a d-dimensional Brownian motion. If $\sigma=\begin{pmatrix}A & 0\\ B & C\end{pmatrix}$, let $\tilde\sigma=\begin{pmatrix}A & 0\\ 0 & I\end{pmatrix}$, where A is $k\times k$. Let $\tilde X=\begin{pmatrix}Y\\W''\end{pmatrix}$.

By the definition of W, we see that $\tilde X$ solves the equation

$$d\tilde X=\tilde\sigma\,dW.$$

Since $a(x)$ depends only on $x_1,\dots,x_k$, the first k coordinates of x, the same is true of $\sigma(x)$ and $A(x)$. It is then easy to see that the law of $\tilde X$ solves the martingale problem for $\tilde a=\begin{pmatrix}A^2 & 0\\ 0 & I\end{pmatrix}$. By Proposition 4.1, the law of $\tilde X=\begin{pmatrix}Y\\W''\end{pmatrix}$ is uniquely determined.
But

$$X_t=\begin{pmatrix}Y_0\\Z_0\end{pmatrix}+\int_0^t\sigma(X_s)\,dW_s,$$

and so

$$\begin{aligned}Z_t&=Z_0+\int_0^t B(Y_s)\,dW_s'+\int_0^t C(Y_s)\,dW_s''\\ &=Z_0+\int_0^t B(Y_s)A^{-1}(Y_s)\,dY_s+\int_0^t C(Y_s)\,dW_s''.\end{aligned}$$

Therefore $Z\in\sigma((Y_s, W_s''),\ s\in[0,\infty))$, and hence we have uniqueness for the law of $X=\begin{pmatrix}Y\\Z\end{pmatrix}$ as well. $\square$

5. Vertex Boundary Points

In this section we consider uniqueness of the martingale problem in the neighborhood of a vertex point. By a change of coordinates, we may assume the vertex point is 0. We assume the matrix of diffusion coefficients a satisfies:

$$a(x) = a(x/|x|), \qquad x \neq 0. \tag{5.1}$$

Let $\{P^x, x \in \mathbb{R}^d\}$ denote a family of solutions to the martingale problem for the matrix a that form a strong Markov family of solutions.

Recall

$$\tau(0) = \inf\{t; |X_t| = 0\}.$$

Let $\tilde{P}^x$ denote the law of the process X_t under P^x killed when first reaching 0. That is,

$$\tilde{P}^x(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = P^x\Big(\bigcap_{1 \leqq i \leqq n} \{X_{t_i} \in A_i, t_i < \tau(0)\}\Big).$$

We assume throughout this section that:

(5.2) $\tilde{P}^x$ is uniquely determined by the matrix a for $x \neq 0$.

By (5.2) we mean that any two solutions to the martingale problem starting at x agree on $F_{\tau(0)}$.

We start with the following elementary estimate.

Lemma 5.1. *There exists a constant c such that for all $r > 0$ and for all x_0 with $|x_0| \leqq r$,*

$$E^{x_0}(\tau(r)) \leqq c r^2.$$

Proof. Apply Ito's lemma to the function $|x|^2$ to get

$$|X_t|^2 - |X_0|^2 = 2 \int_0^t \sum_i X_s^{(i)} \, dX_s^{(i)} + \int_0^t \sum_i a_{ii}(X_s) \, ds.$$

Taking P^{x_0} expectations at the time $t \wedge \tau(r)$ and using the fact that a is strictly elliptic, we have

$$r^2 \geqq E^{x_0} |X_{t \wedge \tau(r)}|^2 \geqq \bar{c} E^{x_0}(t \wedge \tau(r)).$$

Letting $t \to \infty$ completes the proof. □

Recall the following support theorem ([6], Exercise 6.7.5):

Theorem 5.2. *Given any solution to the martingale problem P^x, $\varepsilon > 0$, $t > 0$, and a continuous function $\Psi: [0, t] \to \mathbb{R}^d$ with $\Psi(0) = x$, then:*

$$P^x\big(\sup_{s \leqq t} |X_s - \Psi(s)| < \varepsilon\big) > 0. \quad \square$$

Let $S = \{x: |x| = 1\}$. We define the following transition probability for a Markov chain on S:

(5.3)
$$Q(x, dy) = P^x(X_{\tau(2)}/2 \in dy;\ \tau(2) < \tau(0)) = \tilde{P}^x(X_{\tau(2)}/2 \in dy),\ x, y \in S.$$

We have the following scaling property:

Proposition 5.3. *If*

$$|x| = r,\ P^x(X_{\tau(2r)}/2r \in dy;\ \tau(2r) < \tau(0)) = Q(x/|x|, dy).$$

Proof. Let $Y_t = r^{-1} x_t$ and let $Z_t = X_{t/r^2}$. Using the homogeneity of a it is elementary to check that if $f \in C^2$, then:

$$f(Y_t) - f(Y_0) - \frac{1}{2} \int_0^t r^{-2} \sum_{i,j} a_{ij}(Y_s) \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_s)\, ds$$

is a martingale under P^y for every $y \in \mathbb{R}^d$ and the same is true when Y_t is replaced by Z_t. Then the law of Y_t under $\tilde{P}^x$ satisfies the martingale problem up to time $\tau(0)$ for the matrix $r^{-2} a$ starting at x/r and the law of Z_t under $\tilde{P}^{x/r}$ satisfies the martingale problem up to time $\tau(0)$ for the matrix $r^{-2} a$ starting at x/r. By [6, Theorem 6.5.4] and (5.2), we conclude that the law of Y_t under $\tilde{P}^x$ and the law of Z_t under $\tilde{P}^{x/r}$ are the same if $x \neq 0$. Since hitting distributions are invariant under time changes, for $|x| = r$,

$$\begin{aligned} Q(x/|x|, dy) &= \tilde{P}^{x/r}(x_{\tau(2)}/2 \in dy) \\ &= \tilde{P}^{x/r}(Z_{\tau(2)}/2 \in dy) \\ &= \tilde{P}^x(Y_{\tau(2)}/2 \in dy) \\ &= \tilde{P}^x(X_{\tau(2r)}/2r \in dy). \quad \square \end{aligned}$$

With this Proposition and the strong Markov property, we have immediately that if $|x| = r$,

(5.4)
$$Q^n(x/|x|, dy) = P^x(X_{\tau(r2^n)}/r2^n \in dy;\ \tau(r2^n) < \tau(0)).$$

We are now ready to apply the Krein-Rutman theorem.

Theorem 5.4. *Suppose F and G are bounded continuous functions on S; $G \geq 0$ but not identically 0. Suppose v_n is a sequence of probability measures on S, i.e., $v_n(S) = 1$. Then*

$$\frac{\int Q^n F(x)\, v_n(dx)}{\int Q^n G(x)\, v_n(dx)} \to c(F, G), \quad as \quad n \to \infty$$

where $c(F, G)$ is a constant depending only on Q, F and G and not the sequence v_n.

Proof. First we show that Q is a compact operator, i.e. $Q\{f: f$ continuous on S, $\|f\| \leq 1\}$ is a relatively compact set. Suppose $\|f\| \leq 1$. Fix $x_0 \in S$, let B be the ball of radius $1/2$ about x_0, and let $\sigma = \inf\{t;\ X_t \in \partial B\}$. For $y \in \mathbb{R}^d$, define $H(y) = \tilde{E}^y f(X_{\tau(2)}/2)$. If $x \in B$, then by the continuity of the paths of X_t, $P^x(\sigma < \tau(0) \wedge \tau(2)) = 1$, and so by the strong Markov property, $Qf(x) = \tilde{E}^x H(X_\sigma)$. Since clearly $\|H\| \leqq 1$, by the Krylov-Safonov theorem ([4], Theorem 2 and proof), there exist K and α independent of H and x_0 such that if $|x-x_0|<1/4$,

$$|E^x H(X_\sigma)-E^{x_0} H(X_\sigma)| \leqq K|x-x_0|^\alpha.$$

It follows that $Qf(x)$ is uniformly continuous on S with a modulus of continuity independent of f. It is clear that $\|Qf\| \leqq 1$, and relative compactness follows by the Ascoli-Arzela theorem.

Secondly, we show Q is strongly positive, i.e. if f is continuous, $f \geqq 0$ but not identically 0, then $Qf(x)>0$ for all x. Given such an f, there exists a $y_0 \in S$, a $c>0$ and an $\varepsilon \in (0, 1/2)$ such that $f(y)>c$ whenever $|y-y_0|<\varepsilon$. Fix x, and define ψ: $[0, 3] \to \mathbb{R}^d$ by

$$\psi(s)=\begin{cases}\varphi(s) & 0 \leqq s \leqq 1\\ s y_0 & 1 \leqq s \leqq 3.\end{cases}$$

where φ: $[0, 1] \to S$ is continuous, $\varphi(0)=x$ and $\varphi(1)=y_0$.

Applying Theorem 5.2 with this ε and ψ and $t=3$, we get

$$\begin{aligned} Qf(x) &\geqq \tilde{E}^x(f(X_{\tau(2)}/2;\ |(1/2)X_{\tau(2)}-y_0|<\varepsilon)\\ &\geqq c P^x(\sup_{s \leqq t} |X_s-\psi(s)|<\varepsilon)>0. \end{aligned}$$

We now apply the Krein-Rutman theorem ([3], Theorems 6.1, 6.3, and the proof of Theorem 6.3). So there exists an eigenvalue $\rho \in (0, \infty)$, an eigenfunction φ that is strictly positive and continuous, a functional Φ, and an operator Q_1 such that:

a) if $f \geqq 0$ but f is not identically 0, then $\Phi(f)>0$;

b) $\limsup_{n \to \infty} \|Q_1^n\|^{1/n}<\rho$;

c) Q can be decomposed as:

$$Qf(x)=\rho \Phi(f)\, \varphi(x)+Q_1 f(x) \quad \text{for all continuous } f \text{ and all } x \in S;$$

and moreover

d) $Q^n f(x)=\rho^n \Phi(f)\, \varphi(x)+Q_1^n f(x)$.

It follows that:

$$\|\rho^{-n} Q^n F-\Phi(F)\, \varphi\| \to 0 \quad \text{as} \quad n \to \infty,$$

or integrating with respect to v_n,

$$|\rho^{-n} \textstyle\int Q^n F(x)\, v_n(dx)-\Phi(F) \int \varphi(x)\, v_n(dx)| \to 0 \tag{5.5}$$

with a similar limit holding with F replaced by G. Now

$$\inf_n \Phi(G) \textstyle\int \varphi(x)\, v_n(dx) \geqq \Phi(G) \inf_{x \in S} \varphi(x) \inf_n v_n(S)>0$$

since φ is strictly positive and $v_n(S)=1$ for all n.

We can then take a ratio to get

$$\lim_{n\to\infty} \frac{\int Q^n F(x)\,\nu_n(dx)}{\int Q^n G(x)\,\nu_n(dx)} = \lim_{n\to\infty} \frac{\Phi(F)\int \varphi(x)\,\nu_n(dx)}{\Phi(G)\int \varphi(x)\,\nu_n(dx)} = \frac{\Phi(F)}{\Phi(G)}$$

a constant independent of ν_n. □

We are now ready to prove the main result of this section.

Theorem 5.5. *Under assumptions* (5.1) *and* (5.2), *for each x there is at most one solution P^x to the martingale problem starting at x.*

Proof. We first reduce this problem to a simpler statement. Since for all r and x, $P^x(\tau(r)<\infty)=1$ by Lemma 5.1, it suffices to show uniqueness of P^x up to time $\tau(M)$ for each M. Let $M>2$ be fixed. Since we are considering only strong Markov families of solutions, it suffices to show uniqueness of the operators R_λ, $\lambda>0$, defined by:

$$R_\lambda h(x) = E^x \int_0^{\tau(M)} e^{-\lambda t} h(X_t)\,dt$$

for h bounded and measurable (cf. [6], Corollary 6.2.4). Consider the operator $R(=R_0)$ given by:

$$Rh(x) = E^x \int_0^{\tau(M)} h(X_t)\,dt.$$

Since
$$\sup_{|x|\le M} |Rh(x)| \le \sup_{|x|\le M} E^x[\tau(M)] \sup_{|x|\le M} |h(x)| \le c M^2 \sup_{|x|\le M} |h(x)|,$$

we have that R is a bounded operator on the set of bounded functions whose support lies in the ball of radius M. By [1], Theorem V.5.10, to show uniqueness for the operators R_λ, it suffices to show uniqueness for R.

If $x \neq 0$, by the strong Markov property we have

$$\begin{aligned}
(5.6)\qquad Rh(x) &= E^x \int_0^{\tau(M)} h(X_t)\,dt \\
&= E^x \int_0^{\tau(M)\wedge\tau(0)} h(X_t)\,dt + E^x\left(E^0 \int_0^{\tau(M)} h(X_t)\,dt;\ \tau(0)<\tau(M)\right) \\
&= \tilde{E}^x \int_0^{\tau(M)} h(X_t)\,dt + \tilde{P}^x(\tau(0)<\tau(M))\,E^0 \int_0^{\tau(M)} h(X_t)\,dt.
\end{aligned}$$

Let

$$I(h) = E^0 \int_0^{\tau(M)} h(X_t)\,dt = Rh(0).$$

By (5.6) the value of $Rh(x)$ for $x \neq 0$ is completely determined by $\tilde{P}^x$ and I. So if we show uniqueness for the functional I, we will have uniqueness for the operator R and hence for the family $\{P^x\}$.

Since a is strictly elliptic, then we know $\{t: x_t = 0\}$ has Lebesgue measure 0, a.s. P^0 [9]. Hence by dominated convergence, to show uniqueness for I, it suffices to show uniqueness for $I(h)$ for functions h that are 0 in a neighborhood of 0. Let, then, h be a bounded function such that for some $\delta \in (0, 1)$, $h(x) = 0$ if $|x| \leqq 2\delta$. We will show $I(h)$ depends only on h and $\{\tilde{P}^x, x \neq 0\}$, which with assumption (5.2) will complete the proof.

If $x \neq 0$, let $\psi: [0, t] \to \mathbb{R}^d$ be defined by $\psi(s) = (1+s)x$. Applying Theorem 5.2 with $\varepsilon = |x|/2$, $t = 1 + 2M/|x|$, and the above ψ, we conclude that $P^x(\tau(M) < \tau(0)) > 0$.

If $\varepsilon \leqq \delta$, then by the strong Markov property,

$$\begin{aligned} I(h) &= E^0\left[E^{X_{\tau(\varepsilon)}} \int_0^{\tau(M)} h(X_t)\, dt\right] \\ &= E^0\left[E^{X_{\tau(\varepsilon)}} \int_0^{\tau(M) \wedge \tau(0)} h(X_t)\, dt\right] \\ &\quad + E^0\left[E^{X_{\tau(\varepsilon)}}\left(E^0 \int_0^{\tau(M)} h(X_t)\, dt;\, \tau(0) < \tau(M)\right)\right] \\ &= E^0\left[\tilde{E}^{X_{\tau(\varepsilon)}} \int_0^{\tau(M)} h(X_t)\, dt\right] + I(h)\, E^0[\tilde{P}^{X_{\tau(\varepsilon)}}(\tau(0) < \tau(M))] \end{aligned}$$

and so

$$I(h) = \frac{E^0\left[\tilde{E}^{X_{\tau(\varepsilon)}} \int_0^{\tau(M)} h(X_t)\, dt\right]}{E^0[\tilde{P}^{X_{\tau(\varepsilon)}}(\tau(0) > \tau(M))]}. \tag{5.7}$$

Let us consider the numerator of (5.7). For $x \neq 0$, let $f(x) = \tilde{E}^x \int_0^{\tau(M)} h(X_t)\, dt$, and define $F: S \to \mathbb{R}$ by $F(y) = f(\delta y)$. Let $x_0 \in \mathbb{R}^d$ with $|x_0| = \delta$, let B be the ball of radius $\delta/2$ about x_0, and let $\sigma = \inf\{t;\, X_t \in \partial B\}$. Since $\sigma < \tau(0)$, a.s., by the strong Markov property and the fact that h is 0 inside of B,

$$\begin{aligned} f(x) &= \tilde{E}^x \int_0^{\tau(M)} h(X_t)\, dt \\ &= \tilde{E}^x\, \tilde{E}^{X_\sigma} \int_0^{\tau(M)} h(X_t)\, dt \\ &= \tilde{E}^x f(X_\sigma) \\ &= E^x f(X_\sigma) \end{aligned}$$

for $x \in B$.

Using the Krylov-Safonov theorem as in the proof of Theorem 5.4, f is continuous in a neighborhood of x_0. It follows that F is continuous on S.

Using the strong Markov property and (5.4), if $|x|=2^{-n}\delta$,

$$\begin{aligned} f(x) &= \tilde{E}^x \int_0^{\tau(M)} h(X_s)\,ds = \tilde{E}^x \tilde{E}^{X_{\tau(\delta)}} \int_0^{\tau(M)} h(X_s)\,ds \\ &= \tilde{E}^x f(X_{\tau(\delta)}) = \tilde{E}^x F(X_{\tau(\delta)}/\delta) \\ &= \int F(z)\, P^x(X_{\tau(\delta)}/\delta \in dz;\ \tau(\delta) < \tau(0)) \\ &= Q^n F(x/|x|). \end{aligned}$$

Let v_n be the P^0 distribution of $\delta^{-1} 2^n X_{\tau(2^{-n}\delta)}$. Since $\tau(2^{-n}\delta) < \infty$, a.s., $v_n(S) = 1$. In what follows, v_n will be the only place that the particular solution P^0 of the martingale problem starting from 0 plays a role. Taking $\varepsilon = \delta 2^{-n}$, the numerator of (5.7) is then

$$\begin{aligned} E^0 f(X_{\tau(\varepsilon)}) &= E^0 Q^n F(X_{\tau(\varepsilon)}/\varepsilon) \\ &= \int Q^n F(z)\, v_n(dz). \end{aligned}$$

We treat the denominator of (5.7) similarly.

Let $g(x) = \tilde{P}^x(\tau(0) > \tau(M))$ and let $G\colon S \to \mathbb{R}$ be defined by $G(y) = g(\delta y)$. Letting B and σ be defined as above, $g(x) = E^x g(X_\sigma)$ for $x \in B$. Again using Krylov-Safonov, we conclude that G is continuous on S. Moreover, we have already shown that $g(x) > 0$ if $x \neq 0$, hence $G > 0$.

If $|x| = \delta 2^{-n}$, by the strong Markov property,

$$\begin{aligned} g(x) &= \tilde{P}^x(\tau(0) > \tau(M)) = \tilde{E}^x \tilde{P}^{X_{\tau(\delta)}}(\tau(0) > \tau(M)) \\ &= \tilde{E}^x g(X_{\tau(\delta)}), \end{aligned}$$

and as above, we see that $g(x) = Q^n G(x/|x|)$. The denominator of (5.7) then becomes (with $\varepsilon = 2^{-n}\delta$):

$$\begin{aligned} E^0 \tilde{P}^{X_{\tau(\varepsilon)}}(\tau(0) > \tau(M)) &= E^0 g(X_{\tau(\varepsilon)}) \\ &= \int Q^n G(z)\, v_n(dz). \end{aligned}$$

Now substitute in (5.7), let $n \to \infty$, and apply Theorem 5.4 to obtain:

$$I(h) = \lim_{n \to \infty} \frac{\int Q^n F(y)\, v_n(dy)}{\int Q^n G(y)\, v_n(dy)} = \frac{\Phi(F)}{\Phi(G)} = c(F, G).$$

Finally, observe that f and g, hence F and G, depend on $\{\tilde{P}^x, x \neq 0\}$ and that the kernel Q also depends only on $\{\tilde{P}^x, x \neq 0\}$ and not P^0. Hence $I(h)$ is uniquely determined by $\{\tilde{P}^x, x \neq 0\}$, and the proof is complete. □

Remark 1. Define the operator U on S by

$$U(x, dy) = P^x(2X_{\tau(1/2)} \in dy).$$

Since for $|x|=1$, it is possible that $P^x(\tau(1/2)<\infty)<1$, U need not be conservative. By scaling and the strong Markov property, we have

$$U^n(x, dy)=P^x(2^n X_{\tau(2^{-n})}\in dy;\ \tau(2^{-n})<\infty).$$

By using virtually the same argument as for Q, we get that if F and G are continuous positive functions on S, then:

$$\frac{U^n F(x)}{U^n G(x)} \to \frac{\Gamma(F)}{\Gamma(G)} \tag{5.8}$$

for some positive linear functional Γ on the set of continuous functions on S. By the Riesz representation theorem, there exists a finite measure μ such that $\Gamma(F)=\int F(y)\,\mu(dy)$ for all continuous F.

Let F be continuous on S, let $f(y)=F(y/|y|)$ for $y\neq 0$, let $G=1$, and let $\tilde{\mu}(dy)=\mu(dy)/\mu(S)$. Applying (5.8), we get:

$$E^x[f(X_{\tau(2^{-n})})\,|\,\tau(2^{-n})<\infty]=\frac{U^n F(x)}{U^n G(x)} \to \frac{\int F(y)\,\mu(dy)}{\mu(S)}=\int F(y)\,\tilde{\mu}(dy)$$

as $n\to\infty$. This makes precise the intuitively reasonable statement that the distribution of $2^n X_{\tau(2^{-n})}$ given $\tau(2^{-n})<\infty$ converges weakly to an invariant probability $\tilde{\mu}$, independent of the starting point.

Remark 2. Now define the operator V on S by:

$$V(x, dy)=P^x(X_{\tau(2)}/2\in dy).$$

Here we are not killing the process at 0, so now V is conservative.

Exactly as in the above remark, there exists a probability measure ν such that for $x\neq 0$, $E^x[f(X_{\tau(2^n)})\,|\,\tau(2^n)<\infty]\to\int f(y)\,\nu(dy)$. If $|x|=1$, $P^x(\tau(2^n)<\infty)=1$, and so we in fact have

$$E^x(f(X_{\tau(2^n)}))\to\int f(y)\,\nu(dy)$$

and by scaling,

$$E^{rx/|x|\,2^n} f(X_{\tau(r)})\to\int f(y)\,\nu(dy) \quad \text{as} \quad n\to\infty. \tag{5.9}$$

By the Krylov-Safonov theorem, $E^y f(X_{\tau(r)})$ is continuous as a function of y in a neighborhood of 0, and so taking a limit as $n\to\infty$ in (5.9) gives

$$E^0 f(X_{\tau(r)})=\int f(y)\,\nu(dy).$$

We have thus proved that starting from 0 the hitting distributions $\{X_{\tau(r)}, r\geqq 0\}$ form a stationary process with invariant measure ν.

6. Uniqueness

We now have all the ingredients necessary to prove Theorem 2.1.

Proof of Theorem 2.1. By [6], Theorems 6.4.3 and 10.2.2, it suffices to consider the case where $b(x)=0$. Existence of a solution to the martingale problem follows from [6], Exercise 12.4.3.

To prove uniqueness, we use induction. The case $d=1$ follows by [6], Exercise 7.3.3. So suppose now that the theorem is true for dimensions $1, 2, \ldots, d-1$. We must prove uniqueness when the dimension is d.

By [6], Theorem 6.6.1 and Sect. 7.2, it suffices to show that for every x_0, there exists $\tilde{a}\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ such that $\tilde{a}(x)=a(x)$ for x in some neighborhood of x_0 and there is uniqueness for the martingale problem for $\tilde{a}$ starting at x_0. So suppose first that x_0 is in the interior of one of the polyhedra A_i. Then a is constant in a neighborhood of x_0. If $\tilde{a}(x)$ is defined to be $a(x_0)$, if P is a solution to the martingale problem for $\tilde{a}$ starting at x_0, and if $\tilde{\sigma}$ is the positive definite square root of $\tilde{a}$, then under P, $\tilde{\sigma}^{-1} X_t$ is d-dimensional Brownian motion. Since $\tilde{\sigma}$ is constant, this uniquely characterizes the law of X.

Suppose now that x_0 is a boundary point. Note that there exists $\varepsilon>0$ such that

$$a(x)=a\left(x_0+\frac{\varepsilon(x-x_0)}{|x-x_0|}\right) \quad \text{if } |x-x_0|\leqq\varepsilon.$$

Define

$$\tilde{a}(x)=a\left(x_0+\frac{\varepsilon(x-x_0)}{|x-x_0|}\right) \quad \text{for all } x\in\mathbb{R}^d. \tag{6.1}$$

First suppose that x_0 is a nonvertex boundary point. Then we can choose a coordinate system so that $\tilde{a}(x)$ is a function only of the first k coordinates of x for some $k<d$, and if we write $\tilde{a}=\begin{pmatrix} D & F^* \\ F & G \end{pmatrix}$, then D can be considered as a map from $\mathbb{R}^k$ into $\mathbb{R}^{k\times k}$.

Since $\mathbb{R}^d$ can be divided up into finitely many polyhedra so that $\tilde{a}$ is constant in the interior of each one, geometrical considerations show that this induces a subdivision of $\mathbb{R}^k$ into finitely many polyhedra with D constant in the interior of each one. By the induction hypothesis, we have uniqueness for the martingale problem for D starting from any point. Hence by Theorem 4.3 we have uniqueness for the martingale problem for $\tilde{a}$ starting at x_0.

Finally, suppose x_0 is a vertex boundary point. Note that we can divide $\mathbb{R}^d$ into the union of finitely many polyhedra $\tilde{A}_1, \ldots, \tilde{A}_n$ such that each $\tilde{A}_i$ is a cone with vertex x_0, $\tilde{a}$ is constant in the interior of each $\tilde{A}_i$, and every point of $\mathbb{R}^d-\{x_0\}$ is either in the interior of $\tilde{A}_i$ or a nonvertex boundary point (relative to $\tilde{a}$). By the preceding paragraphs, for each $x\neq x_0$, we have uniqueness of the martingale problem for $\tilde{a}$ starting at x up to the time of first exit from some neighborhood of x.

By a change of coordinate systems, we may assume $x_0=0$. By the method of [6], Sect. 7.2, we have uniqueness for the martingale problem for $\tilde{a}$ starting at $x\in\mathbb{R}^d-\{0\}$ up to time $\tau(1/n)$, and since n is arbitrary, uniqueness up to time $\tau(0)$.

To show uniqueness for $\tilde{a}$ starting at an arbitrary point, it suffices by Exercise 12.4.3 and the proof of Theorem 12.2.4 of [6] to consider only strong Markov families of solutions $\{P^x\}$ (in Exercise 12.4.3 replace the use of 7.3.2 by [9]). We are thus led to the following situation: to show uniqueness of the martingale problem for $\tilde{a}$, we need to show uniqueness for P^x, where $\{P^x\}$ is a family of solutions that form a strong Markov process; each P^x is uniquely determined up to time $\tau(0)$; and by (6.1), $\tilde{a}(x)=\tilde{a}(x/|x|)$ for $x\neq 0$. Applying Theorem 5.5 with $\tilde{a}$ in place of a then completes the proof. $\square$

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