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Rates of convergence to Brownian local time

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Suppose S_n is a mean zero, variance one random walk. Under suitable assumptions on the increments, we prove a strong approximation theorem for the local times of S_n to the local times of a Brownian motion, uniformly at all levels.

1. Introduction

Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean 0 and variance 1. Let S_n be the usual partial sum process. Define the 'local time' of the random walk S_n by

$$\eta(k, n) = \#\{j \leq n: |S_j - k| \leq \tfrac{1}{2}\}.$$

If the X_i's are integer valued, then $\eta(k, n)$ denotes the number of visits of $S_1, \ldots, S_n$ to k. Let Z_t be a standard 1-dimensional Brownian motion and denote its local time by $L(x, t)$. In 1981 Révész [11] proved that if S_n is a simple symmetric random walk, then one could find a probability space supporting a Brownian motion and a simple symmetric random walk such that

$$\sup_{x\in\mathbb{Z}}|\eta(x, n) - L(x, n)| = O(n^{1/4+\varepsilon}) \quad \text{a.s.} \tag{1.1}$$

for any $\varepsilon > 0$. Since Révész's work, there have been a number of papers seeking to improve the rate of convergence and to weaken the assumptions on the X's. See [6] and [3] and the references therein.

The goal of this paper is to obtain what seems to be the optimal rate, under fairly weak assumptions on the X_i's. Let us consider the lattice case first with the X_i's taking values in $\mathbb{Z}$. [6] showed that if X_1 possesses a moment generating function which is finite in a neighborhood of the origin, then the rate in (1.1) can be improved to

$$n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}. \tag{1.2}$$

This rate is achieved by a Skorokhod embedding of S_n in Z_t. They also show that this is the best possible rate for any Skorokhod embedding. We first prove that the above rate (1.2) holds whenever the X_i's have $5+\varepsilon$ moments.

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For the nonlattice case, previous work includes that of [3] who obtained a rate less optimal than (1.2) under the assumption of 8 or more moments. Borodin also required the assumption that

$$\int |\phi(u)|^2 \,\mathrm{d}u < \infty, \tag{1.3}$$

where $\phi(u) = \mathbb{E} \exp(iu\xi_1)$ is the characteristic function of the increments. It is easy to see that Borodin's condition implies that S_2 has a bounded density (see Section 4). We require much less: that for some j_0 the distribution of S_{j_0} has a nonzero absolutely continuous part. We then obtain the rate (1.2) when the X_i's have $6+\varepsilon$ moments.

The bulk of the work is done in Section 2. There we obtain a moment estimate on how much local time at 0 of the Brownian motion increases up until the first visit of the random walk to $[-\frac{1}{2}; \frac{1}{2}]$. Once we have this, we can in Section 3 handle the lattice case quite easily. The necessary modifications for the nonlattice case are done in Section 4; the key idea is the use of the ergodic theorem for an appropriate additive functional.

The letter c, with or without subscripts, denotes constants whose values are unimportant and which may change from line to line.

2. Skorokhod embedding

Let X_i be a sequence of i.i.d. random variables with mean 0, variance 1 and $\mathbb{E}|X_i|^r < \infty$ for some $r \in (2, \infty)$. Let $S_n = \sum_{i=1}^n X_i$. As usual, the random walk is either lattice or nonlattice. In the lattice case, let us assume that the lattice is $\mathbb{Z}$ and the random walk is strongly aperiodic ([12]); we leave to the reader the easy modifications necessary for the general lattice case.

Let Z_t be Brownian motion, and let $\tau(j)$ be a sequence of stopping times embedding the random walk in Z_t. That is, $\tau(0) = 0$, the $\tau(j) - \tau(j-1)$ are i.i.d., and $Z(\tau(j)) - Z(\tau(j-1))$ has the same law as X_j. There is no loss of generality in taking $X_j = Z(\tau(j)) - Z(\tau(j-1))$, and so $S_n = Z(\tau(n))$. We will sometimes write $\mathbb{P}$ for $\mathbb{P}^0$.

In this paper we require that the $\tau(j)$ be the Skorokhod embedding defined in [4]. It is very likely that our results also hold for some of the other Skorokhod embeddings as well.

Let

$$I_j = [2^j, 2^{j+1}], \quad j = 1, 2, \ldots, \qquad I_0 = [0, 2].$$

Let $J = [-\frac{1}{2}, \frac{1}{2}]$. Let

$$T_j = \min\{i \colon |S_i| \in I_j\}, \qquad \sigma = \min\{i \colon S_i \in J\}.$$

We start with some upper bounds on σ. Note that in the next two lemmas, only second moments are necessary in the proofs.

Lemma 2.1. (a) *For each R there exists c such that*

$$\sup_{|x|\leq R} \mathbb{P}^x(\sigma > n) \leq c/\sqrt{n}.$$

(b) *There exists c such that*

$$\sup_{x\in J} \mathbb{P}^x(T_j < \sigma) \leq c2^{-j}.$$

Proof. Let $\rho_{[a,b]} = \min\{i\colon S_i \in [a, b]\}$, so that $\sigma = \rho_{[-1/2,1/2]}$. Suppose $\delta < \frac{1}{2}$. If

$$B_j = \{|S_j| < \tfrac{1}{2}\delta, |S_i| > \tfrac{1}{2}\delta \text{ for } j+1 \leq i \leq n\},$$

then

$$\begin{aligned}\mathbb{P}^0(B_j) &\geq \mathbb{P}^0(|S_j| < \tfrac{1}{2}\delta, |S_i - S_j| > \delta \text{ for } j+1 \leq i \leq n)\\ &= \mathbb{P}^0(|S_j| < \tfrac{1}{2}\delta)\mathbb{P}^0(\rho_{[-\delta,\delta]} > n-j)\\ &\geq \mathbb{P}^0(|S_j| < \tfrac{1}{2}\delta)\mathbb{P}^0(\rho_{[\delta,\delta]} > n).\end{aligned}$$

By the local central limit theorem ([12, Theorem 7.9] in the lattice case, [13] in the nonlattice case), $\mathbb{P}^0(|S_j| < \frac{1}{2}\delta) \geq c/\sqrt{j}$ if j is large enough, c depending on δ. The B_j are disjoint, so for n large enough,

$$1 \geq \mathbb{P}^0\left(\bigcup_{j=[n/2]}^{n} B_j\right) \geq \left(\sum_{j=[n/2]}^{n} c/\sqrt{j}\right)\mathbb{P}^0(\rho_{[-\delta,\delta]} > n) = c\sqrt{n}\mathbb{P}^0(\rho_{[-\delta,\delta]} > n).$$

Let I be any closed interval of length less than $\frac{1}{8}$ contained in $[-R, R]$. If we are in the lattice case, we insist that $I \cap \mathbb{Z} \neq \emptyset$ as well. By the local central limit theorem, for some m and c, $\mathbb{P}^0(S_m \in I) > c$. By taking $\delta < \frac{1}{8}$ small enough, we get, changing m and c is necessary, that

$$\mathbb{P}^0(S_m \in I, S_1, \ldots, S_{m-1} \notin [-\delta, \delta]) \geq c_1.$$

It follows that

$$c/\sqrt{n} \geq \mathbb{P}^0(\rho_{[-\delta,\delta]} > n+m) \geq c_1 \inf_{y\in I} \mathbb{P}^y(\rho_{[-\delta,\delta]} > n).$$

Hence for some $y \in I$, $\mathbb{P}^y(\rho_{[\delta,\delta]} > n) \leq c/\sqrt{n}$, c depending on δ and R. By translation invariance, if $x \in I$,

$$\mathbb{P}^x(\rho_{[y-x-\delta,y-x+\delta]} > n) \leq c/\sqrt{n},$$

c depending on δ and R. Since $|y-x| \leq \frac{1}{8}$ and $\delta < \frac{1}{8}$, then $\rho_{[y-x-\delta,y-x+\delta]} \geq \sigma$, so

$$\mathbb{P}^x(\sigma > n) \leq c/\sqrt{n}. \tag{2.1}$$

This and a covering argument prove (a) for n large. For n small the result is trivial since probabilities are bounded by 1 and we can get our result by taking c large enough.

By the invariance principle

$$\mathbb{P}^0\left(\max_{k\leq 2^{2j}}|S_k|\leq \tfrac{1}{4}\cdot 2^j\right)>c_1.$$

If $T_j<\sigma$ and in the next 2^{2j} steps S_k moves a distance at most $\frac{1}{4}\cdot 2^j$, then $\sigma>2^{2j}$. So

$$c_1\sup_{y\in J}\mathbb{P}^y(T_j<\sigma)\leq \sup_{y\in J}\mathbb{P}^y(\sigma>2^{2j})\leq c2^{-j}$$

by part (a). This gives (b). □

Let

$$N_j=\sum_{i=0}^{\sigma}1_{I_j}(S_i).$$

Lemma 2.2. *There exist c_1 and c_2 such that*

$$\sup_x \mathbb{P}^x(N_j\geq m2^{2j})\leq c_1\exp(-c_2 m).$$

Proof. We prove the result for large j, the case of small j being much easier (cf. proof of Lemma 2.1). Since $\operatorname{Var} X_1=1$, there exist b_1, b_2, and c_1 such that if $y\in J$,

$$\mathbb{P}^y(|S_1|\in[\tfrac{1}{2}+b_1,b_2])\geq c_1. \tag{2.2}$$

Let $\mu=\sigma\wedge c2^{2j}$. If $|z|\leq b_2$, then by Lemma 2.1(a),

$$\mathbb{E}^z\mu\leq\sum_{k=0}^{c2^{2j}}\mathbb{P}^z(\sigma\geq k)\leq 1+c\sum_{k=1}^{c2^{2j}}k^{-1/2}\leq c2^j.$$

If also $|z|\geq\frac{1}{2}+b_1$, then since S_n^2-n and S_n are both martingales,

$$\begin{aligned}b_1+\tfrac{1}{2}&\leq|z|=|\mathbb{E}^z(S_\mu;\mu=\sigma)+\mathbb{E}^z(S_\mu;\sigma>c2^{2j})|\\&\leq\tfrac{1}{2}+(\mathbb{E}^zS_\mu^2)^{1/2}(\mathbb{P}^z(\sigma>c2^{2j}))^{1/2}=\tfrac{1}{2}+(\mathbb{E}^z\mu)^{1/2}(\mathbb{P}^z(\sigma>c2^{2j}))^{1/2},\end{aligned}$$

or $\mathbb{P}^z(\sigma>c2^{2j})\geq b_1^2/c2^j$. With (2.2),

$$\inf_{y\in J}\mathbb{P}^y(\sigma>c2^{2j}+1)\geq c2^{-j}. \tag{2.3}$$

Let $A_i=\{S_i\in J, S_{i+1}\notin J,\dots,S_{c2^{2j}}\notin J\}$. If $|x|\in I_j$, by the local central limit theorem there exists c not depending on j such that

$$\mathbb{P}^x(S_i\in J)\geq c2^{-j}\quad\text{if}\quad 2^{2j}\leq i\leq c2^{2j}.$$

If $|x|\in I_j$, $2^{2j}\leq i\leq c2^{2j}$, then using (2.3)

$$\mathbb{P}^x(A_i)\geq\mathbb{E}^x(\mathbb{P}^{S_i}(\sigma>c2^{2j});\ S_i\in J)\geq\mathbb{P}^x(S_i\in J)\inf_{y\in J}\mathbb{P}^y(\sigma>c2^{2j})\geq c2^{-2j}.$$

Hence, since the A_i are disjoint, for $|x|\in I_j$,

$$\mathbb{P}^x(\sigma<c2^{2j})\geq\mathbb{P}^x\left(\bigcup_{i=2^{2j}}^{c2^{2j}-1}A_i\right)\geq c_2>0.$$

Therefore

$$\sup_x \mathbb{P}^x(N_j \geqslant c2^{2j}) = \sup_{|x| \in I_j} \mathbb{P}^x(N_j \geqslant c2^{2j}) \leqslant \sup_{|x| \in I_j} \mathbb{P}^x(\sigma > c2^{2j}) \leqslant 1 - c_2.$$

Since N_j is a subadditive functional, our result follows immediately. □

Write τ for $\tau(j)$. Let $L(x, t)$ denote the local times for the Brownian motion Z_t.

Lemma 2.3.

$$\sup_{x \in I_j} \mathbb{P}^x(L(0, \tau) > 0) \leqslant c2^{-rj}.$$

Proof. Let Θ be independent of Z_t and uniformly distributed on $\{1, \ldots, N\}$ for some $N \in \mathbb{Z}^+$. Let $U(\theta), D(\theta)$ be nonnegative strictly increasing functions on $1, \ldots, N$. Let $\tau = \inf\{t: Z_t \notin [-D(\Theta), U(\Theta)]\}$, and suppose X has the $\mathbb{P}^0$ law of $Z(\tau)$. We first prove our result for such X with bounds independent of N.

Given $\Theta = \theta$, the probability that Z_t hits $U(\theta)$ before $D(\theta)$ is equal to $D(\theta)/(U(\theta) + D(\theta))$. So

$$\mathbb{E}|X|^r \geqslant \mathbb{E}(X^+)^r = \sum_{u \in \mathrm{Range}(U)} u^r \mathbb{P}(X = u) = \frac{1}{N} \sum_\theta \frac{U(\theta)^r D(\theta)}{U(\theta) + D(\theta)}. \tag{2.4}$$

Similarly,

$$\mathbb{E}|X|^r \geqslant \frac{1}{N} \sum_\theta \frac{D(\theta)^r U(\theta)}{U(\theta) + D(\theta)}. \tag{2.5}$$

Suppose $x \in I_j$. B Chebyshev,

$$\mathbb{P}^x(|X| \geqslant \tfrac{1}{2} \cdot 2^j) \leqslant \mathbb{E}|X|^r / (\tfrac{1}{2} \cdot 2^j)^r \leqslant c2^{-rj}.$$

If $|X| \leqslant \frac{1}{2} \cdot 2^j$ but $X < 0$, then $D(\Theta) \leqslant \frac{1}{2} \cdot 2^j$, and Z_t does not hit 0 before time τ, or $L(0, \tau) = 0$.

The remaining possibility is if $|X| \leqslant \frac{1}{2} \cdot 2^j$ but $X > 0$, and hence $U(\Theta) \leqslant \frac{1}{2} \cdot 2^j$. Now $L(0, \tau) > 0$ only if Z_t hits 0 before time τ, and this is impossible if $D(\Theta) \leqslant 2^j$. If $D(\Theta) > 2^j$, then the probability that Z_t hits 0 before time τ is, conditional on $\Theta = \theta$, less than or equal to $U(\theta)/(2^j + U(\theta))$. Let $A = \{\theta: U(\theta) \leqslant \frac{1}{2} \cdot 2^j, D(\theta) > 2^j\}$. Then

$$\begin{aligned} \mathbb{P}^x(|X| \leqslant \tfrac{1}{2} \cdot 2^j, X > 0, L(0, \tau) > 0) &\leqslant \frac{1}{N} \sum_{\theta \in A} \frac{U(\theta)}{2^j + U(\theta)} \\ &\leqslant \frac{1}{N} \sum_{\theta \in A} \frac{U(\theta)}{2^j}. \end{aligned} \tag{2.6}$$

But by (2.5),

$$\begin{aligned} \mathbb{E}|X|^r &\geqslant \frac{1}{N} \sum_{\theta \in A} \frac{D(\theta)^r U(\theta)}{U(\theta) + D(\theta)} \geqslant \frac{1}{2N} \sum_{\theta \in A} \frac{D(\theta)^r U(\theta)}{D(\theta)} \\ &\geqslant \frac{2^{(r-1)j}}{2N} \sum_{\theta \in A} U(\theta). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7) gives our result in this case.

Any mean 0, variance 1 random variable X can be written as the limit of random variables $X^{(n)}=Z(\tau^{(n)})$ with τ the limit of stopping times $\tau^{(n)}$ of the form described in the first paragraph (cf. [4]). By changing to another probability space if necessary, we can assume $Z(\tau^{(n)})\to Z(\tau)$ a.s. Since $\mathbb{P}^x(L(\tau,0)>0)=\mathbb{P}^x(\inf_{s\leqslant\tau} Z_s<0)$ by the joint continuity of Brownian local time, the lemma follows. □

Let us introduce the terminology that a random variable Y is a *defective exponential* with parameters ρ, R, and we will write $Y\sim \mathrm{DE}(\rho,R)$ if $R>0$, $\rho\in[0,1]$, and

$$\mathbb{P}(Y>x)=\rho\,\mathrm{e}^{-Rx},\quad x>0,\qquad \mathbb{P}(Y=0)=1-\rho.$$

So Y could be considered the product of an independent Bernoulli(ρ) and an exponential(R).

A variation of Lemma 2.3 is:

Lemma 2.4. *If $m<j-1$, set $\rho=2^{-j(r-1)-m}$, $R=2^{-j}$. If $m\geqslant j-1$, set $\rho=2^{-m(r-1)-j}$, $R=2^{-m}$. Then for all $\lambda>0$,*

$$\sup_{x\in I_j}\mathbb{P}^x(L(0,\tau)>\lambda,|X|\in I_m)\leqslant\mathbb{P}(Y>\lambda),$$

where $Y\sim\mathrm{DE}(c\rho,cR)$.

Proof. Recall that if $S=\inf\{t\colon Z_t\notin[-a,b]\}$, then $L(0,S)$ is stochastically smaller than an exponential($a^{-1}\vee b^{-1}$).

To prove Lemma 2.4, we again suppose that X is of the form described in Lemma 2.3 and take limits. There are a number of cases. We will do the hardest one; the others are similar. So suppose $m\geqslant j$, $X<0$. Then $D(\Theta)\in I_m$, and the probability that Z_t hits 0 before time τ is then $\leqslant N^{-1}\sum_{D(\theta)\in I_m}(U(\theta)/(2^j+U(\theta)))$. Let

$$B_1=\{\theta\colon D(\theta)\in I_m,\ U(\theta)\geqslant 2^m\},\qquad B_2=\{\theta\colon D(\theta)\in I_m,\ 2^j\leqslant U(\theta)\leqslant 2^m\},$$

$$B_3=\{\theta\colon D(\theta)\in I_m,\ U(\theta)\leqslant 2^j\}.$$

Now

$$\frac{1}{N}\sum_{B_1\cup B_2}\frac{U(\theta)}{2^j+U(\theta)}\leqslant\frac{1}{N}\#(B_1\cup B_2),$$

while by (2.5)

$$\mathbb{E}|X|^r\geqslant\frac{1}{N}\sum_{B_1}\frac{D(\theta)^rU(\theta)}{U(\theta)+D(\theta)}\geqslant\frac{1}{4N}\sum_{B_1}D(\theta)^r\geqslant\frac{2^{rm}}{4N}\#(B_1)$$

and

$$\mathbb{E}|X|^r\geqslant\frac{1}{N}\sum_{B_2}\frac{D(\theta)^rU(\theta)}{U(\theta)+D(\theta)}\geqslant\frac{1}{4N}\sum_{B_2}D(\theta)^{r-1}U(\theta)\geqslant\frac{2^{(r-1)m+j}}{4N}\#(B_2).$$

On the other hand,

$$\frac{1}{N}\sum_{B_3}\frac{U(\theta)}{2^j+U(\theta)}\leqslant\frac{2^{-j}}{N}\sum_{B_3}U(\theta),$$

while by (2.5) again,

$$\mathbb{E}|X|^r\geqslant\frac{1}{N}\sum_{B_3}\frac{D(\theta)^r U(\theta)}{D(\theta)+U(\theta)}\geqslant\frac{2^{m(r-1)}}{2N}\sum_{B_3}U(\theta).$$

Combining,

$$\frac{1}{N}\sum_{B_1\cup B_2\cup B_3}\frac{U(\theta)}{2^j+U(\theta)}\leqslant 2^{-m(r-1)-j},$$

which proves the assertion concerning ρ for this case.

Using the strong Markov property at the first time Z_t hits 0, $L(0,\tau)$ is stochastically smaller than an exponential with parameter $((U(\Theta)+x)\wedge(D(\Theta)-x))^{-1}$. In the case $m\geqslant j-1$, $X<0$, we have $D(\Theta)\in I_m$, $x\in I_j$, and so $R\leqslant c2^{-m}$. □

Lemma 2.5. *Suppose $\rho\in[0,1]$, $R>0$, and we have random variables E_i and increasing σ-fields $\mathcal{G}_i$ such that E_i is $\mathcal{G}_i$ measurable and the law of E_{i+1} given $\mathcal{G}_i$ is stochastically smaller than a* DE(ρ, R). *Then*

$$\mathbb{P}\left(\sum_{i=1}^n E_i>x\right)\leqslant\exp(-\tfrac{1}{2}Rx+\rho n).$$

Proof. Let $a=\frac{1}{2}R$. Then

$$\mathbb{E}(e^{aE_{i+1}}\mid\mathcal{G}_i)\leqslant(1-\rho)+\frac{\rho R}{R-a}=1+\rho.$$

So

$$\mathbb{E}\left(\exp\left(a\sum_{i=1}^n E_i\right)\right)=\mathbb{E}\left(\exp\left(a\sum_{i=1}^{n-1}E_i\right)\mathbb{E}(\exp(aE_n)\mid\mathcal{G}_{n-1})\right)$$
$$\leqslant(1+\rho)\mathbb{E}\left(\exp\left(a\sum_{i=1}^{n-1}E_i\right)\right).$$

By induction,

$$\mathbb{E}\left(\exp\left(a\sum_{i=1}^n E_i\right)\right)\leqslant(1+\rho)^n\leqslant e^{\rho n}.$$

Finally, Chebyshev's inequality yields

$$\mathbb{P}\left(\sum_{i=1}^n E_i>x\right)\leqslant e^{-ax}\mathbb{E}\exp\left(a\sum_{i=1}^n E_i\right)\leqslant\exp(-\tfrac{1}{2}Rx+\rho n).$$

This completes the proof. □

We are ready for the main theorem of this section. Recalling the Skorokhod embedding given by the $\tau(j)$'s, let

$$\Delta = L(0, \tau(\sigma)).$$

Theorem 2.6. *Suppose* $\mathbb{E}|X|^r < \infty$ *for some* $r \in (3, \infty)$. *Then for each* $\delta > 0$,

$$\sup_{x \in J} \mathbb{E}^x \Delta^{r-1-\delta} < \infty.$$

Proof. Let $K \in \mathbb{Z}^+$. We will obtain an estimate on $\mathbb{P}(\Delta \geq 2^K)$. Let

$$V_j = L(0, \tau(j)) - L(0, \tau(j-1)).$$

Take ε small and let $K_0 = [K/(1+\varepsilon)]$.

First we consider $j \geq K_0$. Let $x \in J$. If ν_l is the lth time that $S_i \in I_j$, then by the strong Markov property and Lemma 2.3,

$$\mathbb{P}^y(S_{\nu_l} \in I_j,\ V_{\nu_l+1} > 0) \leq c2^{-rj}.$$

Then by Lemmas 2.1 and 2.2,

$$\begin{aligned}
&\mathbb{P}^x\left(\sum_{i=0}^{\sigma-1} 1_{I_j}(S_i) V_{i+1} > 0\right)\\
&\quad \leq \mathbb{P}^x(T_j < \sigma)\Bigg[\sup_y \mathbb{P}^y(N_j \geq 2^{(2+2\varepsilon)j})\\
&\qquad\qquad + \sup_y \mathbb{P}^y(S_{\nu_l} \in I_j \text{ and } V_{\nu_l+1} > 0 \text{ for some } l \leq 2^{(2+2\varepsilon)j})\Bigg]\\
&\quad \leq 2^{-j}\left[\exp(-c2^{\varepsilon j}) + 2^{(2+2\varepsilon)j} \sup_y \mathbb{P}^y(S_{\nu_l} \in I_j,\ V_{\nu_l+1} > 0)\right]\\
&\quad \leq 2^{-j}[\exp(-c2^{\varepsilon j}) + c2^{(2+2\varepsilon)j}2^{-rj}]\\
&\quad \leq c2^{j(1-r+2\varepsilon)}.
\end{aligned}$$

We get a similar estimate when we replace I_j by $-I_j$. Summing from K_0 to ∞,

$$\mathbb{P}^x\left(\sum_{i=0}^{\sigma-1} \sum_{j=K_0}^{\infty} 1_{I_j}(|S_i|) V_{i+1} > 0\right) \leq c2^{K_0(1-r+2\varepsilon)}. \tag{2.8}$$

We now consider $m \geq K_0$, $j \leq K_0$. By Chebyshev and the strong Markov property,

$$\sup_y \mathbb{P}^y(|X_{\nu_l+1}| \in I_m) \leq c2^{-m}.$$

So

$$\begin{aligned}
&\mathbb{P}^x\left(\sum_{i=0}^{\sigma-1} 1_{I_j}(S_i) 1_{I_m}(|X_{i+1}|) V_{i+1} > 0\right)\\
&\quad \leq \mathbb{P}^x(T_j < \sigma)\Bigg[\sup_y \mathbb{P}^y(N_j > mK2^{(2+\varepsilon)j})\\
&\qquad\qquad + \sup_y \mathbb{P}^y(|X_{\nu_l+1}| \in I_m \text{ for some } l \leq mK2^{(2+\varepsilon)j})\Bigg]\\
&\quad \leq 2^{-j}[\exp(-c_1 mK2^{\varepsilon j}) + cmK2^{(2+\varepsilon)j}2^{-mr}].
\end{aligned}$$

Summing over m from K_0 to ∞ and doing a similar estimate for $-I_j$, we get

$$\mathbb{P}^x\left(\sum_{i=0}^{\sigma-1}\sum_{m=K_0}^{\infty} 1_{I_j}(|S_i|)1_{I_m}(|X_{i+1}|)V_{i+1}>0\right)\leqslant c2^{K_0(1-r+3\varepsilon)},$$

and since we are considering here the case where $j\leqslant K_0$,

$$\mathbb{P}^x\left(\sum_{i=0}^{\sigma-1}\sum_{m\geqslant K_0, j\leqslant K_0} 1_{I_j}(|S_i|)1_{I_m}(|X_{i+1}|)V_{i+1}>0\right)\leqslant cK_0 2^{K_0(1-r+3\varepsilon)}. \tag{2.9}$$

We now consider j, $m\leqslant K_0$. We will show that in this case

$$\mathbb{P}^x\left(\sum_{i=0}^{\sigma-1} 1_{I_j}(S_i)1_{I_m}(|X_{i+1}|)V_{i+1}>2^K/K^2\right)\leqslant c2^{K_0(1-r+4\varepsilon)}. \tag{2.10}$$

Once we have (2.10), together with a similar bound with I_j replaced by $-I_j$, then summing over the K_0^2 possible values of j and m will give

$$\mathbb{P}^x\left(\sum_{i=0}^{\sigma-1} 1_{I_j}(|S_i|)1_{I_m}(|X_{i+1}|)V_{i+1}>2^K\right)\leqslant K_0^2 c2^{K_0(1-r+4\varepsilon)}. \tag{2.11}$$

Then (2.8), (2.9) and (2.11) together give

$$\sup_{x\in J}\mathbb{P}^x(\Delta>2^K)\leqslant cK^2 2^{K(1-r+4\varepsilon)/(1+\varepsilon)}.$$

Taking ε small enough then gives us the desired estimate on $\mathbb{P}^x(\Delta>2^K)$ to complete the proof.

So we look at (2.11). Suppose $m<j-1$.

$$\begin{aligned}
&\mathbb{P}^x\left(\sum_{i=0}^{\sigma-1} 1_{I_j}(S_i)1_{I_m}(|X_{i+1}|)V_{i+1}>2^K/K^2\right)\\
&\leqslant \mathbb{P}^x(N_j\geqslant cK^2 2^{(2+2\varepsilon)j})\\
&\quad+\mathbb{P}^x\left(\sum_{l=1}^{cK^2 2^{(2+2\varepsilon)j}} 1_{I_j}(S_{\nu_l})1_{I_m}(|X_{\nu_l+1}|)V_{\nu_l+1}>2^K/K^2\right).
\end{aligned} \tag{2.12}$$

By Lemma 2.2, the first term on the right of (2.12) is less than $c\exp(-c_1cK^2 2^{\varepsilon j})\leqslant c\exp(-cK^2)$. Using Lemmas 2.4 and 2.5 with $\rho=2^{-j(r-1)-m}$, $R=2^{-j}$, $x=2^K/K^2$, $n=cK^2 2^{(2+2\varepsilon)j}$, and $\mathcal{G}_l=\sigma(S_{\nu_1},\ldots,S_{\nu_l})$, the second term on the right hand side of (2.12) is

$$\leqslant \exp(-c2^{-j}2^K/K^2+c2^{-j(r-1)-m}K^2 2^{(2+2\varepsilon)j}).$$

Now

$$j\leqslant K_0=[K/(1+\varepsilon)]\leqslant K-\tfrac{1}{2}K\varepsilon,$$

or $K-j\geqslant\frac{1}{2}K\varepsilon$. Since $r>3$, $(2+2\varepsilon)j-j(r-1)-m<0$ if ε is small enough. Thus the second term on the right hand side of (2.12) is

$$\leqslant \exp(-c2^{K\varepsilon/2}/K^2+cK^2)\leqslant c\ \exp(K_0(1-r+4\varepsilon)).$$

This gives (2.13) when $m<j-1$. The case $m\geqslant j-1$ is very similar. □

Corollary 2.7. *For each $\delta > 0$,*

$$\sup_{x \in \mathbb{R}} \mathbb{E}\Delta^{r-2-\delta} < \infty.$$

Proof. The proof is the same as the proof of Theorem 2.6, except that we no longer have the term $\mathbb{P}^x(T_j < \sigma)$ to help us. This accounts for the exponent $r-2-\delta$. $\square$

Remarks. (1) Csörgő and Horváth [6] proved $\mathbb{E}^0\Delta^2 < \infty$ when $\mathbb{E}|X_1|^3 < \infty$ in the lattice case.

(2) Theorem 2.6 is trivial when S_n is a simple symmetric random walk.

3. Lattice case

In this section we assume the X_i are i.i.d., $\mathbb{Z}$-valued and strongly aperiodic. We assume now that $\mathbb{E}|X_i|^r < \infty$ for some $r > 5$.

Let $\sigma(1) = \min\{i > 0\colon S_i = 0\}$, $\sigma(j+1) = \min\{i > \sigma(j)\colon S_i = 0\}$. Let

$$\Delta_i = L(\tau(\sigma(i)), 0) - L(\tau(\sigma(i-1)), 0).$$

By the strong Markov property, the Δ_i are i.i.d., and by Theorem 2.6, have more than 4 moments. Let

$$\eta(x, n) = \sum_{i=0}^{n} 1_{\{x\}}(S_i), \quad x \in \mathbb{Z},\ n \in \mathbb{Z}^+,$$

and define $\eta(x, t)$ by linear interpolation for other values of x and t. Let $\kappa = \mathbb{E}^0\Delta_1$. Later we shall show $\kappa = 1$.

Lemma 3.1. *For $\varepsilon > 0$ sufficiently small,*

$$\mathbb{P}\left(\sup_{i \le m}\left|\sum_{j=1}^{i} \Delta_j - \kappa i\right| > c_1(m \log m)^{1/2}\right) \le cm^{-(1+\varepsilon/8)}.$$

Proof. Let $\bar{\Delta}_i = \Delta_i 1_{(\Delta_i \le m^{1/2-\varepsilon/16})}$. Then

$$\mathbb{P}(\bar{\Delta}_i \ne \Delta_i \text{ for some } i \le m) \le m\mathbb{P}(\bar{\Delta}_1 \ne \Delta_1) = m\mathbb{P}(\Delta_1 \ge m^{(1/2-\varepsilon/16)})$$
$$\le m\frac{\mathbb{E}\Delta_1^{4+\varepsilon/2}}{m^{(1/2-\varepsilon/16)(4+\varepsilon/2)}} \le cm^{(1+\varepsilon/8)}$$

if ε is sufficiently small. Since

$$\mathbb{E}(\Delta_i - \bar{\Delta}_i) \le \int_{m^{1/2-\varepsilon/16}}^{\infty} \mathbb{P}(\Delta_i - \bar{\Delta}_i > x)\,\mathrm{d}x$$
$$\le c\int_{m^{1/2-\varepsilon/16}}^{\infty} x^{-(1/2-\varepsilon/16)(4+\varepsilon/2)}\,\mathrm{d}x \le m^{-1/2-\varepsilon/16},$$

then $\sum_{i=1}^{m}|\mathbb{E}\bar{\Delta}_i - \kappa| = o(m^{1/2})$.

So to prove the lemma, it suffices to show

$$\mathbb{P}\left(\sup_{j\le m}\left|\sum_{i=1}^{j}(\bar{\Delta}_i-\mathbb{E}\bar{\Delta}_i)\right|>c_1(m\log m)^{1/2}\right)\le cm^{-(1+\varepsilon/8)}.$$

But by Bernstein's inequality, since Var $\bar{\Delta}_i\le\mathbb{E}\bar{\Delta}_i^2\le\mathbb{E}\Delta_i^2=\mathbb{E}\Delta_1^2<\infty$ and $\bar{\Delta}_i$ is bounded, this probability is

$$\le\exp\left(\frac{-c_1^2 m\log m}{cm+cm^{1/2-\varepsilon/16}c_1(m\log m)^{1/2}}\right)$$
$$\le\exp(-c_2\log m)\le cm^{-(1+\varepsilon/8)}$$

if c_1 is large enough. □

Let

$$A_n=\left\{\sup_{x\in\mathbb{Z}}\eta(x,m)\le 4(m\log\log m)^{1/2}\text{ for all } m\ge n\right\},$$
$$B_n=\{|\tau(m)-m|\le 4(m\log\log m)^{1/2}\text{ for all } m\ge n\},$$
$$C_n=\{|S_m|\le 4(m\log\log m)^{1/2}\text{ for all } m\ge n,\ |Z_t|\le 4(t\log\log t)^{1/2}\text{ for all } t\ge n\}.$$

Lemma 3.2. $1_{A_n}\to 1$ *a.s.*, $1_{B_n}\to 1$ *a.s.*, $1_{C_n}\to 1$ *a.s.*

Remark. The assertion concerning 1_{A_n} follows from [10]. We give a proof, however, that will also work for the nonlattice case of Section 4.

Proof of Lemma 3.2. Since the X_i have more than 5 moments, then

$$\sup_{t\le 1}|S_{[nt]}/\sqrt{n}-Z_{nt}/\sqrt{n}|=O(n^{-\beta}),\quad\text{a.s.}$$

for some $\beta>0$ (for a proof see [8], for example). Since $L(\sqrt{n}x, nt)/\sqrt{n}$ is the local time at time t of the Brownian motion $Z_{nt}/\sqrt{n}$, then by [1],

$$\sup_{x\in\mathbb{Z}/\sqrt{n},\,t\le 1}|\eta(\sqrt{n}x,[nt])/\sqrt{n}-L(\sqrt{n}x,[nt])/\sqrt{n}|=O(n^{-\beta/12})\quad\text{a.s.}\tag{3.1}$$

By [10],

$$\limsup_n\sup_y\frac{L(y,n)}{(n\log\log n)^{1/2}}=\sqrt{2}\quad\text{a.s.}$$

It follows immediately that

$$\limsup_n\sup_{y\in\mathbb{Z}}\frac{\eta(y,n)}{(n\log\log n)^{1/2}}=\sqrt{2}\quad\text{a.s.,}\tag{3.2}$$

from which $1_{A_n}\to 1$ a.s. follows.

Since Z_t^2-t is a martingale, by the Burkholder-Davis-Gundy inequalities, $\mathbb{E}\tau(1)^2\le c\mathbb{E}|Z(\tau(1))|^4=c\mathbb{E}|X_1|^4<\infty$ and also $\mathbb{E}\tau(1)=\mathbb{E}Z(\tau(1))^2=\mathbb{E}X_1^2=1$. So $1_{B_n}\to 1$ a.s., by the law of the iterated logarithm for the $\tau(i)$ sequence.

The assertion about 1_{C_n} is an immediate consequence of the law of the iterated logarithm for the S_n sequence and the one for Brownian motion. □

Let $r_n = n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}$.

Lemma 3.3.

$$\mathbb{P}\left(\sup_{j\leqslant n}|\eta(0,j)-L(0,j)|\geqslant cr_n;\ A_n\cap B_n\cap C_n\right)\leqslant cn^{-1/2-\varepsilon}.$$

Proof. Let

$$D_m=\left\{\sup_{i\leqslant m}\left|\sum_{j=1}^{i}\Delta_j-\kappa i\right|>c(m\log m)^{1/2}\right\}.$$

Suppose $\omega\in A_n\cap B_n\cap C_n\cap(\bigcap_{\{j:2^j\geqslant m\}}D^c_{2^j})$. Then for k sufficiently large,

$$\sup_{j\leqslant k}\left|\sum_{i=1}^{j}\Delta_i-\kappa j\right|(\omega)<c(k\log k)^{1/2}. \tag{3.3}$$

By (3.1), $\eta(0,m)\to\infty$ as $m\to\infty$. In (3.3), take $j=\eta(0,m)$ and note $\Delta(\eta(0,m))=L(0,\tau(\sigma(\eta(0,m))))$. Since $m\geqslant\sigma(\eta(0,m))$,

$$\begin{aligned}L(0,\tau(m))&\geqslant L(0,\tau(\sigma(\eta(0,m))))=\Delta(\eta(0,m))\\&\geqslant\kappa\eta(0,m)-c(\eta(0,m)\log\eta(0,m))^{1/2}.\end{aligned}$$

Since $m\leqslant\sigma(\eta(0,m)+1)$, setting $j=\eta(0,m)+1$,

$$\begin{aligned}L(0,\tau(m))&\leqslant L(0,\tau(\sigma(\eta(0,m)+1)))=\Delta(\eta(0,m)+1)\\&\leqslant\kappa(\eta(0,m)+1)+c([\eta(0,m)+1]\log[\eta(0,m)+1])^{1/2}\\&\leqslant\kappa\eta(0,m)+c_1(\eta(0,m)\log\eta(0,m))^{1/2}.\end{aligned}$$

Hence for n large,

$$\sup_{j\leqslant n}|L(0,\tau(j))-\kappa\eta(0,j)|\leqslant c(\eta(0,m)\log\eta(0,m))^{1/2}.$$

Since $\omega\in A_n$,

$$\sup_{j\leqslant n}|L(0,\tau(j))-\kappa\eta(0,j)|\leqslant cr_n.$$

By standard estimates on Brownian local time, since $\omega\in B_n$,

$$\sup_{j\leqslant n}|L(0,\tau(j))-L(0,j)|=\mathrm{O}((|\tau(n)-n|\log|\tau(n)-n|)^{1/2})=\mathrm{O}(r_n).$$

Therefore

$$\begin{aligned}&\mathbb{P}\left(\sup_{j\leqslant n}|L(0,j)-\kappa\eta(0,j)|\geqslant cr_n;\ A_n\cap B_n\cap C_n\right)\\&\quad\leqslant\mathbb{P}\left(A_n\cap B_n\cap C_n\cap\left(\bigcup_{2^j\geqslant cn^{1/2+\varepsilon/16}}D_{2^j}\right)\right)\\&\quad\leqslant cn^{-1/2-\varepsilon}\end{aligned}$$

by Lemma 3.1.

Using a standard Borel–Cantelli argument for the sequence $n = 2^i$,

$$\sup_{j \leq n} |\kappa\eta(0, j) - L(0, j)|/\sqrt{n} = \mathrm{O}(r_n/\sqrt{n}) \quad \text{a.s.}$$

Using (3.2) again, we conclude that $\kappa = 1$. □

Theorem 3.4.

$$\sup_{x \in \mathbb{Z}, j \leq n} |\eta(x, j) - L(x, j)| = \mathrm{O}(r_n) \quad a.s.$$

Proof. Fix $y \in \mathbb{Z}$. Let $N = \min\{i \colon Z(\tau(i)) = y\}$, $U = \tau(i)$. By the strong Markov property at time U, Lemma 3.3 tells us that

$$\mathbb{P}\left(\sup_{j \leq n} |[\eta(y, j) - \eta(y, N-1)] - [L(y, j) - L(y, U)]| > cr_n;\ A_n \cap B_n \cap C_n\right)$$
$$\leq cn^{-1/2-\varepsilon}.$$

Of course, $\eta(y, N-1) = 0$, $\mathbb{P}$-a.s. On the other hand, by translation invariance, Chebyshev, and Corollary 2.7,

$$\mathbb{P}(L(y, U) > cr_n) \leq \mathbb{P}^{-y}(\Delta_1 > cr_n) \leq c\mathbb{E}^{-y}\Delta_1^3/r_n^3 \leq n^{-1/2-\varepsilon}.$$

Therefore,

$$\mathbb{P}\left(\sup_{j \leq n} |\eta(y, j) - L(y, j)| > cr_n;\ A_n \cap B_n \cap C_n\right) \leq cn^{-1/2-\varepsilon}. \tag{3.4}$$

Since $\max_{j \leq n} |S_j| \leq n^{1/2+\varepsilon/2}$ and $\sup_{t \leq n} |Z_t| \leq n^{1/2+\varepsilon/2}$ on C_n,

$$\mathbb{P}\left(\sup_{j \leq n} \sup_{y \in \mathbb{Z}} |\eta(y, j) - L(y, j)| > cr_n;\ A_n \cap B_n \cap C_n\right)$$
$$\leq 2n^{1/2+\varepsilon/2} \sup_{y \in \mathbb{Z}, |y| \leq n^{1/2+\varepsilon/2}} \mathbb{P}\left(\sup_{j \leq n} |\eta(y, j) - L(y, j)| > cr_n;\ A_n \cap B_n \cap C_n\right)$$
$$\leq c(n^{1/2+\varepsilon/2})(n^{-1/2-\varepsilon}) \leq cn^{-\varepsilon/4}.$$

We now use Borel–Cantelli along the sequence $n = 2^i$ and Lemma 3.2 to complete the proof. □

Remark. Our method can be modified to give rates for when the X_i have fewer than 5 moments, although the rates will be poorer than (1.2). In this connection, see also [1]. We conjecture that the rate (1.2) holds when the X_i have 4 moments and must deteriorate when the X_i have fewer than 4 moments.

4. Nonlattice case

In this section we obtain the analogous results to Section 3, except we look at the nonlattice case. We assume $\mathbb{E}|X_1|^r < \infty$ for some $r > 6$, and throughout this section we also assume:

Hypothesis 4.1. For some j_0, the law of S_{j_0} has a nonzero absolutely continuous part.

Remark. Borodin [3] uses the condition that $\int |\varphi(u)|^2 \, du < \infty$, where φ is the characteristic function of X_1. By the Fourier inversion formula, this implies that S_2 has a bounded density, and so Hypothesis 4.1 holds in this case.

Let

$$\eta(x, n) = \sum_{i=0}^{n} 1_{[x-1/2, x+1/2]}(S_i),$$

$$\sigma(i) = \min\{j > \sigma(i-1): S_j \in J\}, \quad J = [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Note $Y_i = S_{\sigma(i)}$ is a Markov chain on J.

Recall that X is strongly nonlattice if $\limsup_{|u|\to\infty} |\varphi(u)| < 1$. When this property holds, the results of [1] are applicable. Y_i satisfies Doeblin's condition if there exists a finite measure μ on J, $\varepsilon \in (0, 1)$ and $j \geq 1$ such that if $A \subseteq J$ with $\mu(A) \leq \varepsilon$, then $\sup_{y \in J} \mathbb{P}^y(Y_j \in A) \leq 1 - \varepsilon$.

Lemma 4.2. *If Hypothesis* 4.1. *holds, then*

(a) *X is strongly nonlattice*;

(b) *Y_i satisfies Doeblin's condition.*

Proof. If F is the distribution function of S_{j_0}, we can write $F = \alpha F_a + (1-\alpha) F_s$, $\alpha > 0$, where F_a is the absolutely continuous part of F and F_s is the remainder. Let ψ_a, ψ_s, ψ be the characteristic functions of F_a, F_s, F, respectively. By the Riemann-Lebesgue lemma, $|\psi_a(u)| \to 0$ as $|u| \to \infty$. Thus $\limsup_{|u|\to\infty} |\psi(u)| \leq 1 - \alpha < 1$. Since $\psi(u) = (\varphi(u))^{j_0}$, (a) follows.

By Lemma 2.1, $\sup_{x \in J} \mathbb{P}^x(\sigma(1) > n) \leq c_1/\sqrt{n}$. For any $k > 5$, if $\sigma(k) > nk$, then for at least one $i \leq k$, $\sigma(i+1) - \sigma(i) > n$, and by the strong Markov property,

$$\sup_{x \in J} \mathbb{P}^x(\sigma(k) > nk) \leq c_1 k/\sqrt{n}.$$

Taking $n = c_1^2 k^4$, for any $k > 5$,

$$\sup_{x \in J} \mathbb{P}^x(\sigma(k) > c_1^2 k^5) \leq 1/k < \tfrac{1}{5}. \tag{4.1}$$

Since the convolution of an absolutely continuous distribution with any distribution is absolutely continuous and since the distribution function of S_{kj_0} is

$$(\alpha F_a + (1-\alpha) F_s)^{*k},$$

then the total mass of the singular part of the distribution of S_{kj_0} is $\leq (1-\alpha)^k$. Take $k>5$ large enough so that $(1-\alpha)^k \leq 1/(4c_1^2k^5j_0^5)$, where c_1 is the c_1 of (4.1). Let $j=kj_0$. Note a similar argument shows that the total mass of the singular part of the distribution of S_{j+i} will also be $\leq 1/(4c_1^2j^5)$.

Let μ be Lebesgue measure and note $\alpha(j)\geq j$. Let p_i be the density of the absolutely continuous part of S_i. If $x\in J$, by (4.1),

$$\begin{aligned}\mathbb{P}^x(S_{\sigma(j)}\in A) &\leq \mathbb{P}^x(\sigma(j)>c_1^2j^5)+\sum_{i=j}^{c_1^2j^5}\mathbb{P}^x(S_i\in A)\\ &\leq \tfrac{1}{5}+\sum_{i=j}^{c_1^2j^5}\left[\int_{A-x}p_i(y)\,\mathrm{d}y+(4c_1^2j^5)^{-1}\right]\\ &< \tfrac{1}{5}+\sum_{i=j}^{c_1^2j^5}\int_{A-x}p_i(y)\,\mathrm{d}y+\tfrac{1}{5}. \end{aligned}\tag{4.2}$$

Since $|A-x|=|A|$ for each x and p_i, $i=j,\ldots,c_1^2j^5$ is a finite collection of L^1 functions, then provided $\varepsilon<\frac{1}{4}$ is taken small enough, $\int_{A-x}p_i(y)\,\mathrm{d}y$ will be less than $(4c_1^2j^5)^{-1}$ whenever $|A|<\varepsilon$. Substituting in (4.2),

$$\mathbb{P}^x(S_{\sigma(j)}\in A)<\tfrac{3}{4}\leq 1-\varepsilon,$$

or (b) holds. □

Remarks. (1) Since Doeblin's condition holds, the $\mathbb{P}^x$ law of Y_i converges to some probability measure ν on J exponentially fast, uniformly over $x\in J$ ([7]). Let $F(x)=\mathbb{E}^x\Delta_1$, $\kappa=\int_J F(x)\nu(\mathrm{d}x)$. By Section 2, F is bounded on J.

(2) If the distribution of X_i is purely atomic but nonlattice, it is not hard to see that the random walk is not strongly nonlattice nor do the Y_i satisfy Doeblin's conditions with μ equal to Lebesgue measure.

Lemma 4.3. *If c_1 is large enough,*

$$\sup_{x\in J}\mathbb{P}^x\left(\sup_{j\leq m}\left|\sum_{i=0}^{j}F(S_{\sigma(i)})-\kappa j\right|>c_1(m\log m)^{1/2}\right)\leq cm^{-10}.$$

Proof. We follows a standard argument; see [2], for example. Let

$$G(x)=\mathbb{E}^x\sum_{j=0}^{\infty}[F(S_{\sigma(j)})-\kappa].$$

By Remark 1 immediately preceding, the sum is absolutely convergent and G is bounded. If

$$M_j=G(S_{\sigma(j)})-G(S_0)-\sum_{i=0}^{j}[F(S_{\sigma(i)})-\kappa],$$

then M_j is a martingale with bounded jumps; hence $[M, M]_j \leqslant cj$. So by the martingale version of Bernstein's inequality (cf. [9]), for each K,

$$\mathbb{P}^x\Big(\sup_{j\leqslant m}|M_j| \geqslant c(m \log m)^{1/2}\Big) \leqslant cm^{-K}.$$

Since G is bounded, this proves the lemma. □

Lemma 4.4. *For each $\varepsilon > 0$, if c_1 is large enough.*

$$\sup_{x\in J} \mathbb{P}^x\Big(\sup_{j\leqslant m}\Big|\sum_{i=1}^{j} \Delta_i - \kappa j\Big| > c_1(m \log m)^{1/2}\Big) \leqslant cm^{-(3/2+\varepsilon/8)}.$$

Proof. Write

$$\Delta_i - \kappa = (\Delta_i - F(S_{\sigma(i)})) + (F(S_{\sigma(i)}) - \kappa). \tag{4.3}$$

Lemma 4.3 takes care of the partial sums of the second term on the right of (4.3). Since

$$\mathbb{E}^x(\Delta_{i+1} \,|\, S_{\sigma(1)}, \ldots, S_{\sigma(i)}) = \mathbb{E}^{S_{\sigma(i)}}\Delta_1 = F(S_{\sigma(i)}),$$

then $\sum [\Delta_{i+1} - F(S_{\sigma(i)})]$ is a martingale. So for the first term on the right of (4.3), we proceed as in Lemma 3.1, using the martingale version of Bernstein's inequality and subtracting off the conditional expectations of the truncated random variables. □

Theorem 4.5.

$$\sup_{x\in\mathbb{R}, t\leqslant 1} |\eta(x, [nt]) - L(x, nt)| = \mathrm{O}(r_n) \quad a.s.$$

Proof. Using Lemma 4.4, we proceed exactly as in Section 3 to obtain, if γ is sufficiently small,

$$\sup_{|x|\leqslant n^{1/2+\gamma}, x\in\mathbb{Z}/n^{1/4+\gamma}, t\leqslant 1} |\eta(x, [nt]) - L(x, [nt])| = \mathrm{O}(r_n) \quad \text{a.s.} \tag{4.4}$$

Let

$$\hat{\eta}(x, n) = \sum_{i=0}^{n} 1_{[x-1/2, x+1/2)}(S_i).$$

These are the local times considered in [1]. Since $\hat{\eta}(x, j) - \eta(x, j) = \sum_{i=0}^{j} 1_{\{x+1/2\}}(S_i)$, using Proposition 4.4(b) of [1] it is easy to see that

$$\sup_{j\leqslant n, x\in\mathbb{R}} |\hat{\eta}(x, j) - \eta(x, j)| = \mathrm{o}(r_n) \quad \text{a.s.}$$

If $|x - y| \leqslant 1$, $x < y$, then

$$\eta(x, j) - \eta(y, j) = \sum_{i=0}^{j} (1_{[x-1/2, y-1/2)}(S_i) + 1_{(x+1/2, y+1/2]}(S_i)).$$

So using Proposition 4.4(b) of [1] with $\beta_n = n^{-(1/4+\gamma)}$, we get

$$\sup_{|x-y|\leq n^{-(1/4+\gamma)}, j\leq n} |\eta(x, j) - \eta(y, j)| = o(r_n) \quad \text{a.s.} \tag{4.5}$$

Standard estimates on the modulus of continuity of Brownian local time yield

$$\sup_{|x-y|\leq n^{-(1/4+\gamma)}, j\leq n} |L(x, j) - L(y, j)| = o(r_n) \quad \text{a.s.} \tag{4.6}$$

and

$$\sup_{x\in\mathbb{R}, s\leq n, h\leq 1} |L(x, s+h) - L(x, s)| = o(n^{1/4}) \quad \text{a.s.} \tag{4.7}$$

Now (4.4), (4.5) and (4.6) together give

$$\sup_{|x|\leq n^{1/2+\gamma}, t\leq 1} |\eta(x, [nt]) - L(x, [nt])| = O(r_n) \quad \text{a.s.} \tag{4.8}$$

The result now follows similarly to Section 3 by using Lemma 3.2, (4.7) and (4.8). □

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