Probab. Theory Relat. Fields 91, 307–330 (1992)

Probability Theory and Related Fields

Transition densities for Brownian motion on the Sierpinski carpet

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Received March 21, 1991; in revised form August 16, 1991

Summary. Upper and lower bounds are obtained for the transition densities $p(t, x, y)$ of Brownian motion on the Sierpinski carpet. These are of the same form as those which hold for the Sierpinski gasket. In addition, the joint continuity of $p(t, x, y)$ is proved, the existence of the spectral dimension is established, and the Einstein relation, connecting the spectral dimension, the Hausdorff dimension and the resistance exponent, is shown to hold.

1 Introduction

Let X_t be a Brownian motion whose state space is the Sierpinski carpet F: this is a strong Markov process which has continuous paths and state space F and which is invariant under an appropriate class of transformations. The construction of such processes was one of the principle results of [BB1].

We are naturally interested in the properties of this process, and some results, such as point recurrence and point regularity, were obtained in [BB2]. However to get further information one would really like reasonable estimates for the transition densities (cf. [BP]). The main purpose of this paper is to get good upper and lower bounds for the transition densities $p(t, x, y)$ of X_t and to show continuity in each variable.

There is another way of looking at this problem. Our Brownian motions are constructed as the limit of time-changed reflecting Brownian motions on approximations to the Sierpinski carpet. It is natural to call the infinitesimal generator of the limiting process a Laplacian on F (at present it is not known if there is only one limiting process or several). Estimates on the transition densities of X_t are then just estimates on the fundamental solution to the heat equation on the Sierpinski carpet.

It will be convenient to extend X to the unbounded Sierpinski carpet $\tilde{F} = \bigcup_{k=0}^{\infty} 3^k F$. Let $d_f = \log 8/\log 3$ be the Hausdorff dimension of $\tilde{F}$, and let μ be the multiple of the Hausdorff x^{d_f}-measure on $\tilde{F}$ which assigns mass 1 to F.

Our main result is

★ Research partially supported by NSF Grant DMS 88-22053

Theorem 1.1 *There is a function $p(t, x, y)$, $0 < t < \infty$, $x, y \in \tilde{F}$, such that*
(a) *$p(t, x, y)$ is the transition density of X with respect to μ,*
(b) *$p(t, x, y) = p(t, y, x)$ for all x, y, t.*
(c) *$(t, x, y) \to p(t, x, y)$ is jointly continuous on $(0, \infty) \times \tilde{F} \times \tilde{F}$.*
(d) *There exist constants $c_1, c_2, c_3, c_4 > 0$, and d_w such that, writing $d_s = 2d_f/d_w$,*

$$c_1 t^{-d_s/2} \exp(-c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}) \leqq p(t, x, y) \tag{1.1}$$
$$\leqq c_3 t^{-d_s/2} \exp(-c_4(|x-y|^{d_w}/t)^{1/(d_w-1)}) .$$

(e) *$p(t, x, y)$ is Hölder continuous of order $d_w - d_f$ in x and y and C^∞ in t on $(0, \infty) \times \tilde{F} \times \tilde{F}$. More precisely, there exists a constant c_5 such that*

$$|p(t, x, y) - p(t, x', y)| \leqq c_5 t^{-1} |x - x'|^{d_w - d_f}, \quad \text{for } t > 0, \quad x, x', y \in \tilde{F} ,$$

and for each $k \geqq 1$, $\partial^k p(t, x, y)/\partial t^k$ is Hölder continuous of order $d_w - d_f$ in each space variable.

This is exactly the same form as the estimates obtained in [BP] for the transition density of Brownian motion on the Sierpinski gasket. The only difference is that in the present case the exact value of the constant d_w is unkown—we just have a definition in terms of the limiting resistances of the Sierpinski carpet (see [BBS, BB3]). We show in Sect. 8 that d_s is the 'density of states' for the carpet, or what mathematical physicists call the spectral dimension—see [RT], [W]. We also establish the Einstein relation $d_w = d_f + \tilde{\zeta}$, which connects the Hausdorff and spectral dimensions with the resistance exponent $\tilde{\zeta}$. We may compare the estimate (1.1) with the results in [O] for standard Brownian motion with normal reflection on the 'pre-Sierpinski carpet'.

In fact we will consider not just the standard Sierpinski carpet, but also the other 'carpet like' fractals defined in [BB1]. The techniques of this paper may also be applicable to the study of transition densities on some other classes of fractals, such as the nested fractals defined in [L], but we will not pursue that here.

After some definitions in Sect. 2, we start in Sect. 3 by refining a few results of [BB1, BB2]. We show the Hölder continuity of λ-resolvents in Sect. 4 and then see what information can be obtained from eigenvalue expansions in Sect. 5. In Sect. 6 we prove the upper bound for $p(t, x, y)$, both on and off the diagonal, while the same is done in Sect. 7 for the lower bound. Section 8 contains some further remarks concerning the process: we will see that in many respects our knowledge is as complete as in the case of the Brownian motion on the Sierpinski gasket. The letter c will denote positive, finite constants whose value is unimportant and which may change from one appearance to another. c_i will denote a constant whose value remains fixed within each section of the paper, and depends only on the Sierpinski carpet in question, while $c_{n.i}$ denotes the constant c_i of Section n. Given sequences (a_n), (b_n) we will say $a_n \approx b_n$ if there exists a constant c such that $c^{-1} a_n \leqq b_n \leqq c a_n$.

Some of the results of this paper were announced in [BBS].

2 Notation

We begin by setting up our notation. Let $F_0 = [0, 1]^2$, and let $l \geqq 3$ be fixed. Let $\mathscr{S}_n$ be the collection of closed squares of side l^{-n} with corners in $l^{-n}\mathbb{Z}^2$. Given a set $A \subseteq \mathbb{R}^2$, set

$$\mathscr{S}_n(A) = \{S : S \subset A, S \in \mathscr{S}_n\} .$$

For $S \in \mathscr{S}_n$, let Ψ_S be the orientation preserving linear map which maps F_0 onto S.

We now define a decreasing sequence (F_n) of closed subsets of F_0. Let $R \geqq 1$, and let F_1 be the union of $l^2 - R$ distinct elements of $\mathscr{S}_1(F_0)$. We impose the following conditions on F_1:

(2.1) (H1) (Symmetry) F_1 is preserved by all the isometries which preserve the unit square F_0.

(H2) (Connectedness) $\mathrm{Int}(F_1)$ is connected, and contains a path connecting the lines $\{x_1 = 0\}$ and $\{x_1 = 1\}$.

(H3) (Non-diagonality) The boundary ∂F_1 of F_1 consists of a finite number of disjoint Jordan curves.

(H4) (Borders included) F_1 contains every square in $\mathscr{S}_1$ adjacent to the boundary of F_0.

Remark. These conditions are the ones used in [BB1]. (The list given in (1.1)(iv) of [BB1] should be replaced by the present (H1).) The hypothesis (H4), which was not essential in our previous work, will be used here. (See Sect. 8 for some remarks on how the results of this paper may be modified to cover Sierpinski carpets not satisfying (H4)).

We think of F_1 as being derived from F_0 by removing the interiors of R squares in $\mathscr{S}_1(F_0)$. Given F_1, F_2 is obtained by removing the same pattern from each of the squares in $\mathscr{S}_1(F_1)$. Iterating, we obtain a sequence (F_n), where F_n is the union of $(l^2 - R)^n$ squares in $\mathscr{S}_n(F_0)$. Formally, we define

$$F_{n+1} = \bigcup_{S \in \mathscr{S}_n(F_n)} \Psi_S(F_1) = \bigcup_{S \in \mathscr{S}_1(F_1)} \Psi_S(F_n) .$$

The set $F = \cap F_n$ is a generalized Sierpinski carpet. Let $\tilde{F}_n = \bigcup_{r=0}^{\infty} l^r F_{n+r}$, and $\tilde{F} = \bigcap_{n=0}^{\infty} \tilde{F}_n$. Let also $\partial_a F = \{(x_1, x_2) \in [0, 1]^2 : x_1 = 1 \text{ or } x_2 = 1\}$. Let

$$\mu_n(dx) = (l^2/(l^2 - R))^n 1_{\tilde{F}_n}(x)dx ,$$

and let μ be the weak limit of the $\mu_n : \mu$ is a constant multiple of the Hausdorff x^{d_f}-measure on $\tilde{F}$.

Write W_t^n for Brownian motion on $\tilde{F}_n$ with normal reflection on $\partial \tilde{F}_n$, and set

$$\alpha_n = \sup_{x \in F_n} E^x \tau_n ,$$

where $\tau_n = \inf\{t : W_t^n \in \partial_a F_n\}$.

Let Ω be the collection of continuous paths in $[0, \infty)^2$, and X_t be the canonical coordinate process. Let P_n^x be the law of $W^n(\alpha_n t)$ starting at x, and let $\tau = \inf\{t : X_t \in \partial_a F\}$. One of the main results of [BB1] is the existence of subsequences $n_j \to \infty$ such that for each $x \in F$, the law of $W^{n_j}(\alpha_{n_j}(t \wedge \tau))$ starting at x converges weakly, say to Q^x, and the process (Q^x, X_t) is a continuous strong Markov processes on F.

We wish to study processes on the unbounded carpet $\tilde{F}$. We say that a strong Markov process (P^x, X_t) is a Brownian motion with state space $\tilde{F}$ if there exists a subsequence $n_j \to \infty$ such that for each $x \in \tilde{F}$ the laws $P_{n_j}^x$ converge weakly to P^x.

The existence of such processes follows easily from the results of [BB1]. By Proposition 4.4 of [BB1],

$$\sup_{x \in [0, \frac{1}{2}]^2 \cap F_n} P_n^x(\tau \leqq s) \to 0 \quad \text{as } s \to 0 ,$$

uniformly in n. With this fact, the strong Markov character of W^n, and a diagonalization procedure, it is not hard to show that there exists a subsequence $n_j \to \infty$ such that for all $x \in \tilde{F}$, $P^x_{n_j}$ converges weakly, say to P^x, and (P^x, X_t) forms a strong Markov process on $\tilde{F}$ with continuous paths. As the processes (P^x_n, X_t) are μ_n-symmetric, the limiting process (P^x, X_t) is μ-symmetric. The details of the piecing together argument we leave to the reader.

Remark. One of the main questions left open by our previous work is that of the uniqueness of the Brownian motions (P^x, X_t). While it seems very probable that this process is unique, in principle it is possible that $\{P^x_n, n \geqq 1\}$ could have more than one cluster point. However this lack of uniqueness will not prevent us from being able to say a great deal about the behavior of the Brownian motions on $\tilde{F}$. Let

$$m_F = l^2 - R ;$$

following [L] we will refer to l and m_F as the *length* and *mass scale factors of* F. By Theorem 5.1 of [BB3], there exists a constant $\rho_F > 1$ (*the resistance scale factor of* F) such that

$$\frac{1}{4}\rho_F^n \leqq R_n \leqq 4\rho_F^n , \tag{2.2}$$

where R_n is the resistance of F_n, defined by

$$R_n^{-1} = \inf\left\{ \int_{F_n} |\nabla u|^2(x)dx \colon u \text{ is a function on } F_n \text{ with } u(0, x_2) \equiv 0, u(1, x_2) \equiv 1 \right\} .$$

We define the *time scale factor* of F by

$$t_F = m_F \cdot \rho_F , \tag{2.3}$$

and the 'dimensions' d_f, d_w, d_s by

$$\begin{aligned} d_f &= \log m_F / \log l , \\ d_w &= \log t_F / \log l = \log(m_F \rho_F)/\log l , \\ d_s &= 2d_f/d_w = 2\log m_F/\log t_F . \end{aligned} \tag{2.4}$$

Of these d_f is the Hausdorff dimension of F, and we will see in Sect. 8 that d_s is the spectral dimension of the Sierpinski carpet. Although exact values of d_s and d_w are not known, the numerical calculations in [BBS] indicate that for the standard Sierpinski carpet $\rho_F \simeq 1.2515$, $t_F \simeq 10.012$, $d_w \simeq 2.097$, and $d_s \simeq 1.805$.

Proposition 5.2 of [BB3] implies that $t_F/l^2 \geqq 1$. In fact strict inequality holds, as is clear on inspecting the last but one line of the proof. Therefore we have $m_F < l^2 < t_F$, so that

$$d_s < d_f < 2 < d_w. \tag{2.5}$$

In [BB2] we restricted our attention to the standard Sierpinski carpet. However the results there extend without difficulty to general carpets satisfying (H1)–(H4), and the formulae there remain valid if one replaces the length and mass scale factors of the standard SC (3 and 8 respectively) by l and m_F. In particular, from (2.2), the definition of μ_n, and (2.11) and Propositions 2.2 and 2.3 of [BB2], we have

$$c_1^{-1}(t_F/l^2)^n \leqq \alpha_n \leqq c_1(t_F/l^2)^n . \tag{2.6}$$

Let

(2.7) $$\beta_n^r = \alpha_n l^{2r}/\alpha_{n-r}\,;$$

then by (2.6)

(2.8) $$c_1^{-2} t_F^r \leqq \beta_n^r \leqq c_1^2 t_F^r\,.$$

Let $B(x, a) = \{y: |x - y| < a\}$, and for $x = (x_1, x_2)$, define

$$D_r(x) = \left[\frac{i-1}{l^r}, \frac{i+1}{l^r}\right) \times \left[\frac{j-1}{l^r}, \frac{j+1}{l^r}\right) \quad \text{if} \quad \frac{i-\frac{1}{2}}{l^r} \leqq x_1 < \frac{i+\frac{1}{2}}{l^r},$$

$$\frac{j-\frac{1}{2}}{l^r} \leqq x_2 < \frac{j+\frac{1}{2}}{l^r}, \quad i, j \text{ integers.}$$

Note that $m_F^{-r} \leqq \mu(D_r(x)) \leqq 4m_F^{-r}$ if $x \in \tilde{F}$, and that

(2.9) $$B(x, \tfrac{1}{2} l^{-r}) \subset D_r(x) \subset B(y, 3l^{-r}) \quad \text{for any } y \in D_r(x).$$

Define

(2.10) $$\sigma_r(x) = \sigma_r^X(x) = \inf\{t: X_t \notin D_r(x)\}$$

$$T_x = \inf\{t > 0: X_t = x\}\,.$$

3 Preliminary estimates

Unlike the processes on finitely ramified fractals studies in [BP], [L], we do not have an exact scaling property for X. The following 'weak scaling' result will, however, prove very useful.

Lemma 3.1 *There is a constant c_1, independent of r, and constants $\beta_r \in [c_1^{-2} t_F^r, c_1^2 t_F^r]$, $r \in \mathbb{Z}$, such that if Q^x is equal to the $P^{l^r x}$ law of $l^{-r} X(t\beta_r)$, then (Q^x, X_t) is also a Brownian motion on $\tilde{F}$.*

Remark. It seems very likely that $Q^x = P^x$, that is, that X has an exact scaling property. This would follow immediately if we knew that (P^x, X_t) was unique.

Proof. There exists a sequence $n_j \to \infty$ such that for each $x \in \tilde{F}$, P^x is the weak limit of $P_{n_j}^x$. Using (2.8), we may take a further subsequence, also denoted n_j, such that $\beta_{n_j+r}^r$ converges; call the limit β_r. Take $n_1 + r \geqq 0$.

By the continuity of the paths of X and the choice of the subsequence n_j, the law of $l^{-r} W^{n_j}(t\alpha_{n_j} \beta_{n_j+r}^r)$ starting at $l^r x$ converges to the $P^{l^r x}$ law of $l^{-r} X(t\beta_r)$, which we are calling Q^x. But the law starting at $l^r x$ of $l^{-r} W^{n_j}(t\alpha_{n_j} \beta_{n_j+r}^r) = l^{-r} W^{n_j}(t l^{2r} \alpha_{n_j+r})$ is equal, by Brownian scaling, to the law of $W^{n_j+r}(\alpha_{n_j+r} t)$ starting at x. So Q^x is the weak limit of the law of $W^{m_j}(\alpha_{m_j} t)$ starting at x, where $m_j = n_j + r$ is independent of x. By our definition, Q^x is a Brownian motion on $\tilde{F}$. □

Lemma 3.2 *Let (P^x, X_t) be a Brownian motion on $\tilde{F}$, and let (n_j) be a subsequence such that $P^x = \lim_j P_{n_j}^x$.*
(a) *The laws $P_{n_j}^x(\sigma_r(y) \in .)$ converge weakly to $P^x(\sigma_r(y) \in .)$, $r \in \mathbb{Z}$.*
(b) $\lim_{j \to \infty} E_{n_j}^x \sigma_r(y) = E^x \sigma_r(y)$, $r \geqq 1$.

(c) *There exists a constant* c_2 *such that*

(3.1) $$c_2^{-1} t_F^{-r} \leqq E^x \sigma_r(x) \leqq c_2 t_F^{-r}, \quad r \geqq 1.$$

Proof. (a) Using the Skorohod theorem, we may set up a single probability space $(\Omega, \mathscr{F}, P)$ carrying processes X^j and X with laws $P^x_{n_j}$, and P^x, and with $X^j \to X$ a.s. and uniformly on compacts. Writing $\sigma_r^j = \sigma_r^{X^j}(y)$, we have $\sigma_r \leqq \liminf_j \sigma_r^j$. Using the strong Markov property of X at σ_r, and the invariance of X under local isometries of $\tilde{F}$, it follows that X hits both $\operatorname{int}(D_r(y))$ and $\operatorname{int}(D_r^c(y))$ immediately after σ_r; and hence that $\sigma_r = \lim_j \sigma_r^j$.

(b) By the homogeneity of $\tilde{F}$ and equations (4.2), (4.9), and (4.10) of [BB1] we have for any $x \in D_n(y)$

$$E_n^x \sigma_r^j(y) \leqq \alpha_{n-r}/\alpha_n l^{2r} = 1/\beta_n^r \leqq c_1^2 t_F^{-r} .$$

Thus $P^x(\sigma_r^j(y) > 2c_1^2 t_F^{-r}) \leqq \frac{1}{2}$, and hence by an elementary iteration argument $P^x(\sigma_r^j(y) > 2nc_1^2 t_F^{-r}) \leqq 2^{-n}$. So (σ_r^j) is bounded in L^2, and hence converges in L^1 as well as a.s.

(c) The upper bound is immediate from (b), while the lower bound follows from (b) and equations (4.2), (4.9) and (4.10) of [BB1]. □

Remark. This lemma helps to explain the meaning of the time scale factor t_F: the mean time to cross $l^n F$ is approximately t_F^n times the mean time to cross F.

Proposition 3.3 *There exist* c_3 *and* $c_4 > 0$ *such that*

$$P^x(\sigma_0(x) \leqq s) \leqq c_3 \exp(-c_4 s^{-1/(d_w - 1)}), \quad x \in F .$$

Proof. Look at the proof of Proposition 4.4 of [BB1], and note that, by the homogeneity of $\tilde{F}$ this applies to $\sigma_0(x)$ as well as to τ. Starting in the middle of p. 243 of [BB1] we have

$$P_n^x(\sigma_0(x) \leqq s) \leqq \exp\left(2\left(\frac{k^{2r} m_r \alpha_n s}{\alpha_{n-r} c_6}\right)^{1/2} - m_r \log c_6^{-1} \right),$$

where $k = l$ and $m_r = \frac{1}{2} l^r - 2$. By (2.6)

$$P_n^x(\sigma_0(x) \leqq s) \leqq \exp(c_7((l t_F)^r s)^{1/2} - c_8 l^r), \quad 3 \leqq r \leqq n .$$

Let

$$r = [\log(c_8^2/4c_7^2 s)/\log(t_F/l)] .$$

Since $t_F > l$, we can find c_9 sufficiently small so that whenever $s \leqq c_9$, then $r = r(s) \geqq 3$. With this choice of r, we get

(3.2) $$\begin{aligned} P_n^x(\sigma_0(x) \leqq s) &\leqq \exp(-c_8 l^r/2) , \\ &\leqq \exp(-c_4 s^{-1/(d_w - 1)}), \quad n \geqq r, \quad s \leqq c_9 . \end{aligned}$$

Letting $n \to \infty$ along the subsequence and using Lemma 3.2(a), we obtain (3.2) for P^x. Since we always have $P^x(\sigma_0(x) \leqq s) \leqq 1$, we can find c_3 such that

$$P^x(\sigma_0(x) \leqq s) \leqq c_3 \exp(-c_4 s^{-1/(d_w - 1)}), \quad s \geqq 0 ,$$

which is our result. □

Remark. The exponent in this bound is the correct one: see Corollary 3.3 of [BP] for the corresponding result for Brownian motion on the Sierpinski gasket. That bound was obtained from a detailed study of a branching process associated with the diffusion, and it is, at first, surprising that the apparently crude argument of Proposition 4.4 of [BB1] yields essentially as good a result. The explanation is that the small tail in the distribution of σ_0 is due to those paths which move directly from x to the boundary, so that (in the proof of Proposition 4.4) replacing N by $\frac{1}{2}k^r$ gives away very little. In fact the methods of [BB1] have been used in [H] to obtain estimates on the lower tail of the limiting random variable of a branching process in a random environment.

Theorem 3.4 (a) *There exist c_3 and $c_4 > 0$ such that for any $r \in \mathbb{Z}$*

$$P^x(\sigma_r(x) \leqq t) \leqq c_3 \exp(-c_4(t_F^r t)^{-1/(d_w-1)}) .$$

(b) *For any $\lambda > 0$*

$$P^x\left(\sup_{s \leqq t} |X_s - X_0| > \lambda\right) \leqq c_3 \exp(-c_{11}(\lambda^{d_w}/t)^{1/(d_w-1)}) .$$

Proof. (a) By Lemma 3.1, and writing $y = xl^r$,

$$P^x(\sigma_r(x) \leqq t) = Q^y(\sigma_0(y) \leqq t/\beta_{-r}) ,$$

where $Q^\cdot$ is another Brownian motion of $\tilde{F}$. Now use Proposition 3.3 and the bounds on β_r from (2.8).

(b) Choose $r = [-\log \lambda/\log l]$. By (2.9) $D_r(x) \subseteq B(x, c_{10}\lambda)$ for a constant c_{10}. Hence

$$P^x\left(\sup_{s \leqq t} |X_s - X_0| > c_{10}\lambda\right) \leqq P^x(\sigma_r(x) \leqq t) .$$

Using the bound from (a) and then replacing λ by $c_{10}^{-1}\lambda$ completes the proof. □

Remark. Using weak scaling just as in the proof of Theorem 3.4(a), we see (3.1) is valid for $r \leqq 0$ as well.

Let R_λ be an independent negative exponential random variable with mean λ^{-1}.

Corollary 3.5 *There exist $c_{12}, c_{13} > 0$ such that*

(a) $$P^x(R_\lambda \geqq \sigma_r(x)) \leqq \tfrac{1}{2} \quad \textit{for } \lambda \geqq c_{12} t_F^r,$$

(b) $$P^x(R_\lambda \leqq \sigma_r(x)) \leqq \tfrac{1}{2} \quad \textit{for } \lambda \leqq c_{13} t_F^r.$$

Proof. (a) By Theorem 3.4,

$$\begin{aligned} P^x(R_\lambda \geqq \sigma_r(x)) &= \int_0^\infty \lambda e^{-\lambda t} P^x(\sigma_r(x) \leqq t)dt \\ &\leqq c_3 \int_0^\infty \lambda e^{-\lambda t} \exp(-c_4(tt_F^r)^{-1/(d_w-1)})dt \\ &= c_3 \int_0^\infty e^{-s} \exp(-c_4(st_F^r/\lambda)^{-1/(d_w-1)})ds \\ &= I(u) \quad \text{say,} \end{aligned}$$

where $u = (\lambda/t_F^r)^{1/(d_w-1)}$. Clearly $I(u) \downarrow 0$ as $u \to \infty$, so there exists c_{14} such that $I(u) \leqq \frac{1}{2}$ for $u \geqq c_{14}$, and thus $P^x(R_\lambda \geqq \sigma_r(x)) \leqq \frac{1}{2}$ for $\lambda \geqq t_F^r c_{14}^{d_w-1}$.

The proof of (b) is very similar. As $P^x(\sigma_r(x) \geqq t) \leqq t^{-1} E^x \sigma_r(x) \leqq t^{-1} c_2 t_F^{-r}$, we have

$$P^x(R_\lambda \leqq \sigma_r(x)) = \int_0^\infty \lambda e^{-\lambda t} P^x(\sigma_r(x) \geqq t) dt$$

$$\leqq \int_0^\infty \lambda e^{-\lambda t}(1 \wedge (t^{-1} c_2 t_F^{-r})) dt$$

$$= \int_0^\infty e^{-s}(1 \wedge (s^{-1} c_2 \lambda t_F^{-r})) ds.$$

This last integral decreases to 0 as $(c_2 \lambda t_F^{-r}) \downarrow 0$, so it follows that there exists $c_{13} > 0$ such that $P^x(R_\lambda \leqq \sigma_r(x)) \leqq \frac{1}{2}$ for $\lambda \leqq c_{13} t_F^r$. □

Let L_t^y denote the local time of X_t at the point y. In [BB2] this was constructed for the process $X_{\cdot \wedge \tau}$, but we may extend it to the process X_t by a straightforward patching argument. We have that $(t, y) \to L_t^y$ is jointly continuous, and that L satisfies the density of occupation time formula

$$\int_0^t f(X_s) ds = \int_{\tilde{F}} f(y) L_t^y \mu(dy) . \tag{3.3}$$

Proposition 3.6 *Let $x \in \tilde{F}$. Then*

$$E^z L_{\sigma_m(x)}^y \leqq c_{18} (l^{-m})^{(d_w - d_f)} \quad z, y \in D_m(x) .$$

Proof. Suppose first that $m \geqq 5$, and let $r \geqq m$. They by [BB2], Eq. (3.10), if $n \geqq r$,

$$E_n^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s) ds \leqq c_{15} l^{-2p} \alpha_{n-p} \alpha_n^{-1} l^{pd_f} \mu(A) ,$$

where $p = r - 3$. In [BB2] this is proved when $D_r(x) \subseteq [0, l^{-r+1})^2$, but just as in the remarks following (3.11) of [BB2] we can use the local homogeneity of $\tilde{F}$ to remove this restriction.

From (2.7) we deduce

$$E_n^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s) ds \leqq c_{16} (\beta_n^r)^{-1} l^{rd_f} \mu(A) \leqq c_{17} (m_F/t_F)^r \mu(A) . \tag{3.4}$$

We let $n \to \infty$ along the subsequence n_j and obtain

$$E^y \int_A (L_{\sigma_r(x)}^z - L_{\sigma_{r+1}(x)}^z \mu(dz) = E^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s) ds$$

$$\leqq c_{17} (m_F/t_F)^r \mu(A) .$$

Since L_t^z is jointly continuous in t and z, setting $A = B(x, \varepsilon) \cap \tilde{F}$, dividing both sides by $\mu(A)$, and letting $\varepsilon \downarrow 0$ gives

$$E^y L_{\sigma_r(x)}^x - E^y L_{\sigma_{r+1}(x)}^x \leqq c_{17} (m_F/t_F)^r .$$

Since $m_F < l^2 < t_F$, summing over r gives

$$E^y L^x_{\sigma_m(x)} \leqq \sum_{r=m}^{\infty} c_{17}(m_F/t_F)^r = c_{18}(m_F/t_F)^m = c_{18}(l^{-m})^{(d_w - d_f)} .$$

This takes care of the case $m \geqq 5$. A weak scaling argument as in the proof of Theorem 3.4(a) gives the case $m < 5$. □

4 Resolvents

Since we are working with $\tilde{F}$ instead of F, we will need to work with λ-resolvents rather than Green functions. The next set of results concerns the continuity of λ-resolvents.

Define for $A \subseteq \mathbb{R}^2$,

$$R_A = \inf\{t \geqq 0 : X_t \in A^c\} ,$$

and set for $\lambda \geqq 0$

$$u^\lambda_A(x, y) = E^x \int_0^{R_A} e^{-\lambda s} dL^y_s = E^x L^y_{R_A \wedge R_\lambda} ,$$

$$U^\lambda_A f(x) = E^x \int_0^{R_A} e^{-\lambda s} f(X_s) ds, \quad f \geqq 0 .$$

By the density of occupation time formula (3.3) we have

$$U^\lambda_A f(x) = \int_A u^\lambda_A(x, y) f(y) \mu(dy) .$$

Write

$$u_A(x, y) = u^0_A(x, y), \quad u^\lambda(x, y) = u^\lambda_{\tilde{F}}(x, y) ,$$

and define U_A, U^λ similarly. As (P^x, X_t) is μ-symmetric we have $\mu^\lambda_A(x, y) = u^\lambda_A(y, x)$ for all $x, y \in \tilde{F}$. Note that

(4.1) $$u^\lambda_A(x, y) = P^x(T_y < R_A \wedge R_\lambda) u^\lambda_A(y, y) \leqq u^\lambda_A(y, y) .$$

Lemma 4.1 *For $x \in \tilde{F}$, $r \in \mathbb{Z}$,*

$$c_1^{-1}(m_F/t_F)^r \leqq u_{D_r(x)}(x, x) \leqq c_1(m_F/t_F)^r .$$

Proof. The upper bound is immediate from Proposition 3.6. For the lower bound, writing $A = D_r(x)$, we have by Lemma 3.2(c) and the remark following the proof of Theorem 3.4,

$$c_2^{-1} t_F^{-r} \leqq E^x \sigma_r(x) = E^x \int_A L^y_{\sigma_r(x)} \mu(dy) = \int_A u_A(x, y) \mu(dy)$$

$$\leqq \int_A u_A(x, x) \mu(dy) \leqq c m_F^{-r} u_A(x, x) .$$ □

Approximating $B(x, a)$ inside and outside by sets of the form $D_r(x)$ we deduce

Corollary 4.2 *Let $x \in \tilde{F}$, and $a > 0$. Then*

$$c_3 a^{d_w - d_f} \leqq u_{B(x,a)}(x, x) \leqq c_4 a^{d_w - d_f} .$$

Lemma 4.3 *Suppose $A \subseteq B \subseteq \tilde{F}$, A is bounded, and $\sup_x u_B^\lambda(x, y) < \infty$. For $x, y \in \tilde{F}$ we have*

(4.2) $u_A(x, y) = u_B^\lambda(x, y) + E^x(1_{(R_\lambda \leqq R_A)} u_A(X_{R_\lambda}, y)) - E^x(1_{(R_\lambda > R_A)} u_B^\lambda(X_{R_A}, y))$.

Proof. From the definition of u_A, and as $R_A \leqq R_B$,

$$\begin{aligned} u_A(x, y) &= E^x(L^y_{R_A}; R_\lambda \leqq R_A) + E^x(L^y_{R_A}; R_\lambda > R_A) \\ &= E^x(L^y_{R_\lambda \wedge R_B}; R_\lambda \leqq R_A) + E^x(1_{(R_\lambda \leqq R_A)} E^{X_{R_\lambda}} L^y_{R_A}) \\ &\quad + E^x(L^y_{R_\lambda \wedge R_B}; R_\lambda > R_A) - E^x(L^y_{R_\lambda \wedge R_B} - L^y_{R_A}; R_\lambda > R_A) \\ &= u_B^\lambda(x, y) + E^x(1_{(R_\lambda \leqq R_A)} u_A(X_{R_\lambda}, y)) - E^x(1_{(R_\lambda > R_A)} u_B^\lambda(X_{R_A}, y)). \quad \square \end{aligned}$$

Proposition 4.4 *There exists $c_5 > 0$ such that for all $\lambda > 0$, $x, y \in \tilde{F}$,*

$$c_5^{-1} \lambda^{\frac{1}{2} d_s - 1} \leqq \sup_y u^\lambda(x, y) = u^\lambda(x, x) \leqq c_5 \lambda^{\frac{1}{2} d_s - 1}.$$

Proof. The middle equality is immediate from (4.1). For the upper bound, let $x \in \tilde{F}$ and $\lambda > 0$ be fixed. Choose r so that $c_{3.12} t_F^{r+1} > \lambda > c_{3.12} t_F^r$. Let $A = D_r(x)$, and let $B = D_m(x)$ for some $m < r$. Note first that, by Lemma 4.1,

$$u_B^\lambda(x, x) \leqq u_B(x, x) \leqq c_1 (m_F/t_F)^m < \infty.$$

By Lemma 4.3

$$\begin{aligned} u_B^\lambda(x, x) &\leqq u_A(x, x) + E^x(1_{(R_\lambda > R_A)} u_B^\lambda(X_{R_A}, x)) \\ &\leqq u_A(x, x) + P^x(R_\lambda > R_A) u_B^\lambda(x, x). \end{aligned}$$

As $R_A = \sigma_r(x)$ using Corollary 3.5 we deduce that

$$u_B^\lambda(x, x) \leqq 2u_A(x, x) \leqq 2c_1 (m_F/t_F)^r \leqq c_5 \lambda^{\frac{1}{2} d_s - 1}.$$

Letting $m \downarrow -\infty$ concludes the proof.

The proof of the lower bound is very similar. Let r be chosen so that $c_{3.13} t_F^{r-1} < \lambda \leqq c_{3.13} t_F^r$, and let $A = D_r(x)$. By Lemma 4.3 with $B = \tilde{F}$

$$\begin{aligned} u_A(x, x) &\leqq u^\lambda(x, x) + P^x(R_\lambda \leqq R_A) u_A(x, x) \\ &\leqq u^\lambda(x, x) + \tfrac{1}{2} u_A(x, x), \end{aligned}$$

and the result now follows from Lemma 4.1. $\square$

For $x, y \in \tilde{F}$ set

$$q_A^\lambda(x, y) = P^x(T_y > R_A \wedge R_\lambda), \quad p_A^\lambda(x, y) = P^x(T_y \leqq R_A \wedge R_\lambda)$$

and note that

$$u_A^\lambda(x, y) = p_A^\lambda(x, y) u_A^\lambda(y, y).$$

As in the case of u we write p_A, q_A for p_A^0, q_A^0.

Lemma 4.5 *Let $y \in \tilde{F}$, $n > r$, $D_r(y) \cap \tilde{F} \subseteq A$, and $x \in D_{n-1}(y) - D_n(y)$. Then there exists $c_8 > c_1^{-1}$ such that*

$$q_A(x, y) \leqq c_8 (m_F/t_F)^n / u_A(y, y).$$

Proof. Write $B = D_{n-1}(y) - D_n(y)$, and note that (H4) implies that B is connected. As $q_A(\cdot, y)$ is harmonic on $A - \{y\}$ it follows from the Harnack inequality (Theorem 3.1 of [BB1]) that there exists c_6, independent of y, r and n such that

$$c_6^{-1} \leqq \frac{q_A(z, y)}{q_A(z', y)} \leqq c_6 \quad \text{for } z, z' \in cl(B) ,$$

where $cl(B)$ is the closure of B. Now writing $T = R_{D_n(y)}$,

$$\begin{aligned} u_A(y, y) &= E^y L_T^y + E^y E^{X_T} L_{R_A}^y \\ &= u_{D_n(y)}(y, y) + E^y(1 - q_A(X_T, y))u_A(y, y) . \end{aligned}$$

So $u_{D_n(y)}(y, y) = u_A(y, y)E^y q_A(X_T, y)$, and hence

$$\begin{aligned} q_A(x, y) &\leqq c_6 E^y q_A(X_T, y) \\ &= c_6 u_{D_n(y)}(y, y)/u_A(y, y) . \end{aligned}$$

Applying Lemma 4.1, the result follows. □

Write $\theta = d_w - d_f$.

Lemma 4.6 *Let* $x, y \in \tilde{F}$. *Then*

$$c_9|x - y|^\theta \leqq E^x L_{T_y}^x \leqq c_{12}|x - y|^\theta .$$

Proof. The lower bound is immediate from Corollary 4.2 since $T_y \geqq R_{B(x,|x-y|)}$. For the upper bound choose n such that $x \in D_{n-1}(y) - D_n(y)$, and as $c_1 c_8 \geqq 1$ we can choose a negative integer p such that $c_1 c_8 l^{\theta p} < \frac{1}{2} \leqq c_1 c_8 l^{\theta(p+1)}$. Let $r = n + p$, let $m < r < n$ and set $A = D_r(y)$, $B = D_m(y)$. Then

$$\begin{aligned} E^x L_{T_y \wedge R_B}^x &= E^x(L_{T_y}^x; T_y \leqq R_A) + E^x(L_{T_y \wedge R_B}^x; T_y > R_A) \\ &\leqq E^x L_{R_A}^x + E^x(1_{(T_y > R_A)} E^{X_{R_A}} L_{T_y \wedge R_B}^x) \\ &\leqq u_A(x, x) + q_A(x, y) E^x(L_{T_y \wedge R_B}^x) . \end{aligned}$$

Now by Lemmas 4.1 and 4.5, $q_A(y, x) \leqq c_1 c_8 l^{\theta(r-n)} < \frac{1}{2}$. As $l^{-r} \leqq c_{10}|x - y|$, and $A \subset B(y, c_{11}|x - y|)$,

$$E^x L_{T_y \wedge R_B}^x \leqq 2u_A(x, x) \leqq c_{12}|x - y|^\theta.$$

Letting $m \to -\infty$ concludes the proof. □

Corollary 4.7 *Let* $x \in A$, $A \subsetneq \tilde{F}$. *Then writing* $b = \mathrm{dist}(x, A^c \cap \tilde{F})$,

$$c_{13}^{-1} b^\theta \leqq u_A(x, x) \leqq c_{13} b^\theta .$$

Proof. Let $z \in A^c \cap \tilde{F}$ with $|x - z| < 2b$. Then $R_A \leqq T_z$, so $u_A(x, x) \leqq E^x L_{T_z}^x \leqq c_{12} 2^\theta b^\theta$, proving the upper bound. On the other hand $B(x, b/2) \cap \tilde{F} \subset A$, so $u_A(x, x) \geqq u_{B(x, b/2)}(x, x) \geqq c_3 2^{-\theta} b^\theta$. □

Proposition 4.8 *Let* $x, y \in \tilde{F}$, $A \subsetneq \tilde{F}$. *Then*

$$u_A(y, y) - u_A(x, y) \leqq c_{16}|x - y|^\theta .$$

Proof. Choose n so that $x \in D_{n-1}(y) - D_n(y)$; thus $c_{14}^{-1} l^{-n} \leqq |x-y| \leqq c_{14} l^{-n}$. Set $B = (D_{n-1}(y) - D_n(y)) \cap \tilde{F}$. If $B \subseteq A$ then the result follows from Lemma 4.5 and the equation

$$u_A(y, y) - u_A(x, y) = q_A(x, y) u_A(y, y) .$$

Otherwise we must have $\operatorname{dist}(y, A^c \cap \tilde{F}) < c_{15}|x-y|$, so $u_A(y, y) - u_A(x, y) \leqq u_A(y, y) \leqq c_{13} c_{15}^{\theta} |x-y|^{\theta}$. □

Theorem 4.9 (a) *Let* $\lambda \geqq 0$, $A \subseteq \tilde{F}$ *and suppose that either* $\lambda > 0$ *or* $A \neq \tilde{F}$. *Then for* $x, x', y \in \tilde{F}$, *and* $f \in L^1(\tilde{F})$

$$\text{(4.3)} \qquad |u_A^\lambda(x, y) - u_A^\lambda(x', y)| \leqq c_{18} |x - x'|^{d_w - d_f},$$

$$\text{(4.4)} \qquad |U_A^\lambda f(x) - U_A^\lambda f(x')| \leqq c_{18} |x - x'|^{d_w - d_f} \| f 1_A \|_1 .$$

(b) *For* $\lambda > 0$, $A \subseteq \tilde{F}$, $x, x' \in \tilde{F}$, $f \in L^\infty(\tilde{F})$,

$$\text{(4.5)} \qquad |U_A^\lambda f(x) - U_A^\lambda f(x')| \leqq c_{19} \mu \lambda^{-\frac{1}{2} d_s} |x - x'|^{d_w - d_f} \| f \|_\infty .$$

Proof. (a) Assume for the moment that $A \neq \tilde{F}$. As the process X is symmetric with respect to μ, we have

$$\begin{aligned} u_A^\lambda(x, y) - u_A^\lambda(x', y) &= u_A^\lambda(y, x) - u_A^\lambda(y, x') \\ &= (p_A^\lambda(y, x) - p_A^\lambda(y, x')) u_A^\lambda(x', x') + p_A^\lambda(y, x)(u_A^\lambda(x, x) - u_A^\lambda(x', x')). \end{aligned}$$

However

$$p_A^\lambda(y, x) - p_A^\lambda(y, x') \leqq P^y(T_x \leqq R_A \wedge R_\lambda < T_{x'}) \leqq p_A^\lambda(y, x) q_A^\lambda(x, x') .$$

So,

$$\begin{aligned} \text{(4.6)} \qquad u_A^\lambda(x, y) - u_A^\lambda(x', y) &\leqq p_A^\lambda(y, x)(q_A^\lambda(x, x') u_A^\lambda(x', x') + u_A^\lambda(x, x) - u_A^\lambda(x', x')) \\ &= p_A^\lambda(y, x)(u_A^\lambda(x, x) - u_A^\lambda(x, x')). \end{aligned}$$

Setting $\lambda = 0$ and using Proposition 4.8 we deduce

$$u_A(x, y) - u_A(x', y) \leqq u_A(x, x) - u_A(x, x') \leqq c_{16} |x - x'|^{\theta} .$$

Interchanging x and x' gives (4.3) in the case $\lambda = 0$. Integrating we have,

$$\begin{aligned} \text{(4.7)} \qquad |U_A f(x) - U_A f(x')| &\leqq \int_A |u_A(x, y) - u_A(x', y)| \, |f(y)| \mu(dy) \\ &\leqq c_{16} |x - x'|^{d_w - d_f} \| f 1_A \|_1 , \end{aligned}$$

which establishes (4.4) for $\lambda = 0$.
To obtain estimates for $\lambda > 0$ we apply the resolvent equation in the form

$$u_A^\lambda(x, y) = u_A(x, y) - \lambda U_A g(x),$$

where $g(x) = u_A^\lambda(x, y)$. Thus

$$\begin{aligned} |u_A^\lambda(x, y) - u_A^\lambda(x', y)| &\leqq |u_A(x, y) - u_A(x', y)| + \lambda |U_A g(x) - U_A g(x')| \\ &\leqq c_{16} |x - x'|^{d_w - d_f} + \lambda c_{16} |x - x'|^{d_w - d_f} \| g \|_1 . \end{aligned}$$

Now as

$$\|g\|_1 = \int u_A^\lambda(x, y)\mu(dx) \leqq \int_{\tilde{F}} u^\lambda(x, y)\mu(dx) = \lambda^{-1} ,$$

this implies (4.3), and exactly as in the case $\lambda = 0$, (4.4) follows on integrating (4.3). Finally the case $A = \tilde{F}$, $\lambda > 0$ follows on letting $A \uparrow \tilde{F}$.

(b) Note first that

$$p_A^\lambda(y, x) = P^y(T_x \leqq R_A \wedge R_\lambda) \leqq P^y(T_x \leqq R_\lambda) = u^\lambda(y, x)/u^\lambda(x, x) .$$

So by Proposition 4.4

$$\text{(4.8)} \qquad \begin{aligned} \int_A p_A^\lambda(y, x)|f(y)|\mu(dy) &\leqq \|f\|_\infty u^\lambda(x, x)^{-1} \int_A u^\lambda(y, x)\mu(dy) \\ &= \|f\|_\infty u^\lambda(x, x)^{-1}\lambda^{-1} \\ &\leqq c_5\|f\|_\infty \lambda^{-\frac{1}{2}d_s}. \end{aligned}$$

Combining (4.3) and (4.6) we have

$$\text{(4.9)} \qquad |u_A^\lambda(x, y) - u_A^\lambda(x', y)| \leqq c_{18}(p_A^\lambda(y, x) + p_A^\lambda(y, x'))|x - x'|^\theta ,$$

and the proof of (4.5) is completed by integrating (4.9) and using (4.8). □

5 Eigenvalue expansions

Our next set of results concerns eigenvalue expansions. Fix $x_0 \in \tilde{F}$ and $r \in \mathbb{Z}$, write $A = D_r(x_0) \cap \tilde{F}$ and for this section only write $\bar{u}^\lambda(x, y)$ for $u_{D_r(x_0)}^\lambda(x, y)$, and similarly for $\bar{U}^\lambda f$. Write (f, g) for the inner product on $L^2(A, \mu)$.

Fix for the moment $\lambda > 0$. In view of Proposition 4.4 and Theorem 4.9, we may use the Mercer expansion theorem (see, e.g., [Y] p. 136) to obtain a nonincreasing sequence of reals $\gamma_j > 0$ and an orthonormal sequence of function φ_j in $L^2(A, \mu)$ such that

$$\text{(5.1)} \qquad \bar{u}^\lambda(x, y) = \sum_{j=1}^{\infty} \gamma_j \varphi_j(x)\varphi_j(y) ,$$

$$\text{(5.2)} \qquad \bar{U}^\lambda f(x) = \sum_{j=1}^{\infty} \gamma_j(f, \varphi_j)\varphi_j(x), \quad f \in L^2(A, \mu) .$$

The sums in (5.1) and (5.2) converge uniformly as well as in L^2. Set $\lambda_j = \gamma_j^{-1} - \lambda$, so that $\gamma_j = (\lambda + \lambda_j)^{-1}$. Define

$$\text{(5.3)} \qquad \bar{p}(t, x, y) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y), \quad x, y \in D_r(x_0) \cap \tilde{F} .$$

Proposition 5.1 *The λ_j are strictly positive, and the φ_j are Hölder continuous on A. For $\beta > 0$ we have*

$$\text{(5.4)} \qquad \bar{u}^\beta(x, y) = \sum_{j=1}^{\infty} (\beta + \lambda_j)^{-1}\varphi_j(x)\varphi_j(y) .$$

The series in (5.3) and (5.4) converge absolutely and uniformly on A. $\bar{p}(t, x, y)$ is a version of the transition density for (P^x, X_t) killed on exiting A, and is jointly continuous in (t, x, y) on $(0, \infty) \times \tilde{F} \times \tilde{F}$.

Proof. As the φ_j are orthonormal, $(\lambda + \lambda_j)^{-1}\varphi_j = \bar{U}^\lambda \varphi_j$, and Theorem 4.9 implies that φ_j is bounded and Hölder continuous of order $d_w - d_f$. Next, as $\lambda \bar{U}^\lambda$ is a contraction on $L^2(A, \mu)$, Cauchy-Schwarz gives

$$(\lambda + \lambda_j)^{-1} = \gamma_j = (\bar{U}^\lambda \varphi_j, \varphi_j) \leqq \| \bar{U}^\lambda \varphi_j \|_2 \| \varphi_j \|_2 \leqq \lambda^{-1} .$$

Hence each $\lambda_j \geqq 0$. Further, by Lemma 3.2(c) and the remark following Theorem 3.4, $\sup_x E^x \sigma_r(x_0) < \infty$, and thus

$$\sup_x E^x(1 - \exp(-\lambda \sigma_r(x_0))) < 1 .$$

Therefore

$$\sup_x \lambda \bar{U}^\lambda \varphi_j(x) = \sup_x E^x \int_0^{\sigma_r(x_0)} \lambda e^{-\lambda t} \varphi_j(X_t) dt < \| \varphi_j \|_\infty ,$$

which implies that $\lambda_j > 0$.

For $\beta > 0$ write $\hat{u}^\beta(x, y)$ for the right hand side of (5.4). Since $(\beta + \lambda_j)^{-1} \leqq c(\lambda + \lambda_j)^{-1}$, then for $1 \leqq n \leqq m \leqq \infty$, by Cauchy-Schwarz,

$$\text{(5.5)} \quad \left| \sum_{j=n}^m (\beta + \lambda_j)^{-1} \varphi_j(x) \varphi_j(y) \right|^2 \leqq \left(\sum_n^m (\beta + \lambda_j)^{-1} \varphi_j^2(x) \right) \left(\sum_n^m (\beta + \lambda_j)^{-1} \varphi_j^2(y) \right)$$
$$\leqq \left(c \sum_n^m (\lambda + \lambda_j)^{-1} \varphi_j^2(x) \right) \left(c \sum_n^m (\lambda + \lambda_j)^{-1} \varphi_j^2(y) \right) .$$

Thus $\hat{u}^\beta(x, y)^2 \leqq c u^\lambda(x, x) u^\lambda(y, y) \leqq c \lambda^{1 - d_s/2}$, and the series (5.4) converges absolutely. Further, as the convergence in (5.4) is uniform, the estimate (5.5) shows that the series in (5.4) converges uniformly, so that $\hat{u}^\beta(x, y)$ is continuous on $A \times A$. For $0 < \beta < 2\lambda$ we have

$$\sum_{i=0}^\infty (\lambda - \beta)^i (\lambda + \lambda_j)^{-(i+1)} = (\beta + \lambda_j)^{-1},$$

and hence using (5.2)

$$\text{(5.6)} \qquad \hat{U}^\beta f = \sum_{i=0}^\infty (\lambda - \beta)^i (\bar{U}^\lambda)^{i+1} f, \quad f \in L^2(A, \mu), \quad 0 < \beta < 2\lambda .$$

The resolvent equation implies that $\bar{U}^\beta$ also satisfies (5.6), so $\bar{U}^\beta = \hat{U}^\beta$ for $\beta \in (0, 2\lambda)$, and hence for $\beta \in (0, \infty)$. Finally, as $\hat{u}^\beta$ and $\bar{u}^\beta$ are both continuous, $\hat{u}^\beta = \bar{u}^\beta$.

Since $\exp(-\lambda_j t) \leqq c(t)(\lambda + \lambda_j)^{-1}$, a similar argument shows that the series defining $\bar{p}(t, x, y)$ converges uniformly, and absolutely for each x and y, and is therefore continuous in (t, x, y). As

$$\int_0^\infty e^{-\beta t} \bar{p}(t, x, y) dt = \sum_j (\beta + \lambda_j)^{-1} \varphi_j(x) \varphi_j(y) = \bar{u}^\beta(x, y)$$

for β close to λ, we see that $\bar{p}(t, x, y)$ is a version of the transition density for X_t killed on leaving A. □

Remark. By using the Krein-Rutman theorem [KR] one can actually show $\lambda_1 < \lambda_2$ and $\varphi_1 > 0$.

Lemma 5.2 (a) *$\bar{p}(t, x, y)$ is nonincreasing in t.*
(b) *$\bar{p}(t, x, y) \leqq \bar{p}(t, x, x)^{1/2}\bar{p}(t, y, y)^{1/2}$ for $t > 0$, $x, y \in A$.*
(c) *There exists $c_1 > 0$ (independent of r) such that*

$$\sup_{x,y} \bar{p}(t, x, y) \leqq c_1 t^{-d_s/2} .$$

Proof. (a) is immediate from the definition of $\bar{p}$, while (b) follows immediately from (5.3) by Cauchy-Schwarz. By (b) it is enough to prove (c) in the case $x = y$. Since $p_A(s, x, x)$ is decreasing in s,

$$u^\alpha(x, x) \geqq u^\alpha_A(x, x) = \int_0^\infty e^{-\alpha s} p_A(s, x, x) ds \geqq p_A(t, x, x)\alpha^{-1}(1 - e^{-\alpha t}) .$$

Setting $\alpha = t^{-1}$, and using Proposition 4.4, we have

$$p_A(t, x, x) \leqq c_2 \alpha u^\alpha(x, x) \leqq c_2 c_{4.5} t^{-d_s/2} .$$ □

Theorem 5.3 *$\bar{p}(t, x, y)$ is Hölder continuous: there exists c_3 independent of r such that*

(5.7) $$|\bar{p}(t, x, y) - \bar{p}(t, x', y)| \leqq c_3 t^{-1} |x - x'|^{d_w - d_f} ,$$

with a similar bound for $\bar{p}(t, x, y) - \bar{p}(t, x, y')$.

Proof. Fix t and y, and set

$$R(x) = \sum_{j=1}^\infty (\lambda + \lambda_j) e^{-\lambda_j t} \varphi_j(x) \varphi_j(y) .$$

Note that

(5.8) $$\sup_{a \geqq 0} (\lambda + a) e^{-at/2} \leqq \lambda \vee 2t^{-1} .$$

So by Cauchy-Schwarz and Lemma 5.2

$$\begin{aligned} |R(x)| &\leqq \left(\sum (\lambda + \lambda_j) e^{-\lambda_j t} \varphi_j^2(x) \right)^{1/2} \left(\sum (\lambda + \lambda_j) e^{-\lambda_j t} \varphi_j^2(y) \right)^{1/2} \\ &\leqq \left((\lambda \vee 2t^{-1}) \sum e^{-\lambda_j t/2} \varphi_j^2(x) \right)^{1/2} \left((\lambda \vee 2t^{-1}) \sum e^{-\lambda_j t/2} \varphi_j^2(y) \right)^{1/2} \\ &= (\lambda \vee 2t^{-1}) \bar{p}(t/2, x, x)^{1/2} \bar{p}(t/2, y, y)^{1/2} \\ &\leqq c_1 \lambda (1 \vee 2(\lambda t)^{-1}) t^{-d_s/2} . \end{aligned}$$

On the other hand,

$$\bar{U}^\lambda R(x) = \sum (\lambda + \lambda_j) e^{-\lambda_j t} (\bar{U}^\lambda \varphi_j(x)) \varphi_j(y) = \bar{p}(t, x, y) ,$$

so by Theorem 4.9 we deduce that

$$|\bar{p}(t, x, y) - \bar{p}(t, x', y)| \leqq c_4 |x - x'|^{d_w - d_f} (\lambda t)^{-d_s/2} \lambda (1 \vee 2(\lambda t)^{-1}) .$$

Setting $\lambda = t^{-1}$ gives (5.7), while the Hölder continuity in y is immediate from the symmetry of $\bar{p}(t, x, y)$. □

Proposition 5.4 (a) *For* $k \geqq 1$, $\partial^k \bar{p}(t, x, y)/\partial t^k$ *is Hölder continuous in* x *and* y *with modulus of continuity independent of* r. *In particular,* $\bar{p}(t, x, y)$ *is* C^∞ *in* t.
(b) *For* $k \geqq 1$ *there exist constants* c_k, *depending only on* k, *such that*

$$|\partial^k \bar{p}(t, x, y)/\partial t^k| \leqq c_k t^{-k-d_s/2}, \quad t > 0, \quad x, y \in A . \tag{5.9}$$

(c) *For* $x, y \in A$, $s, t > 0$,

$$|\partial^k \bar{p}(t, x, y)/\partial t^k - \partial^k \bar{p}(s, x, y)/\partial t^k| \leqq c_k |t - s| (s \wedge t)^{-k-d_s/2}.$$

Proof. (a) Note that if

$$S_k(x) = \sum (-\lambda_j)^k (\lambda + \lambda_j) e^{-\lambda_j t} \varphi_j(x) \varphi_j(y) ,$$

then

$$\partial^k \bar{p}(t, x, y)/\partial t^k = \sum_j (-\lambda_j)^k e^{-\lambda_j t} \varphi_j(x) \varphi_j(y) = \bar{U}^\lambda S_k(x) . \tag{5.10}$$

Since $\sup_{a \geqq 0} a^k (\lambda + a) e^{-at/2} = c(k, t) < \infty$, an argument similar to Theorem 5.3 shows that $\partial^k \bar{p}(t, x, y)/\partial t^k$ is Hölder continuous in x and y.

(b) It is easy to check that $a^k e^{-at/2} \leqq (2k)^k t^{-k}$. The bound (5.9) now follows from (5.10) and Lemma 5.2.

(c) This is immediate from (5.9). □

Upper bounds

Let $p_A(t, x, y)$ denote the transition density for (P^x, X_t) killed on exiting A. Fix $x_0 \in \tilde{F}$. From Lemma 5.2 and Theorem 5.3, we see that $\{p_{D_r(x_0)}(t, x, y); r \leqq 0\}$ is equicontinuous in x and y on $\tilde{F}$. Clearly $p_{D_r(x_0)}(t, x, y)$ increases as r decreases. Let us define

$$p(t, x, y) = \lim_{r \to -\infty} p_{D_r(x_0)}(t, x, y) .$$

We then have

Theorem 6.1 (a) *For each* x, $p(t, x, x)$ *is decreasing in* t.
(b) $p(t, x, y) \leqq (p(t, x, x))^{1/2} (p(t, y, y))^{1/2}$.
(c) $p(t, x, y)$ *is symmetric in* x *and* y.
(d) $p(t, x, y) \leqq c_1 t^{-d_s/2}$.
(e) $p(t, x, y)$ *is jointly continuous in* (x, y, t), *and*

$$|p(t, x, y) - p(t, x', y)| \leqq c_2 t^{-1} |x - x'|^{d_w - d_f} ,$$

$$|p(t, x, y) - p(s, x, y)| \leqq c_3 (s \wedge t)^{-1-d_s/2} .$$

(f) $p(t, x, y)$ *is a version of the transition density of* (P^x, X_t) *with respect to* μ.
(g) $p(t, x, y)$ *is* C^∞ *in* t *and* $\partial^k p(t, x, y)/\partial t^k$ *is Hölder continuous of order* $d_w - d_f$ *in each space variable.*

Proof. (a)–(e) follow immediately from the corresponding results for $p_{D_r(x_0)}(t, x, y)$ on letting $r \to -\infty$. For (f) we have, by monotone convergence

$$\begin{aligned} P^x(X_t \in B(y, \varepsilon)) &= \lim_{m \to -\infty} P^x(X_t \in B(y, \varepsilon), R_{D_m(x_0)} > t) \\ &= \lim_{m \to -\infty} \int_{B(y,\varepsilon)} p_{D_m(x_0)}(t, x, z)\mu(dz) \\ &= \int_{B(y,\varepsilon)} p(t, x, z)\mu(dz) . \end{aligned}$$

Now divide by $\mu(B(y, \varepsilon))$, let $\varepsilon \to 0$, and use (e).

Write $q_r(k, t, x, y)$ for $\partial^k p_{D_r(x_0)}(t, x, y)/\partial t^k$. By Proposition 5.4, for each k the $q_r(k, t, x, y)$ are equicontinuous on compact subsets of $(0, \infty) \times \tilde{F} \times \tilde{F}$. So to prove (g) it suffices to show $\lim_{r \to -\infty} q_r(k, t, x, y) = \partial^k p(t, x, y)/\partial t^k$. We do this by induction on k. Fix x, y. The case $k = 0$ is just the definition of $p(t, x, y)$. If $q^{(i)}(k, t, x, y)$ is the limit along any subsequence $r_j^{(i)}$, $i = 1, 2$, then by the induction hypothesis

$$\begin{aligned} \int_a^b q^{(i)}(k+1, t, x, y)dt &= \lim_{r_j^{(i)} \to -\infty} \int_a^b q_{r_j^{(i)}}(k+1, t, x, y)dt \\ &= \lim_{r_j^{(i)} \to -\infty} [q_{r_j^{(i)}}(k, b, x, y) - q_{r_j^{(i)}}(k, a, x, y)] \\ &= q(k, b, x, y) - q(k, a, x, y) \end{aligned} \tag{6.1}$$

for any $0 < a < b, i = 1, 2$. This shows $q^{(1)}(k+1, t, x, y) = q^{(2)}(k+1, t, x, y)$ for each t. So $q_r(k+1, t, x, y)$ converges as $r \to -\infty$, and (6.1) shows that the limit must be $q(k+1, t, x, y)$. □

We now want to get a better bound on $p(t, x, y)$ when $x \neq y$.

Theorem 6.2 *There exist c_4, c_5 such that*

$$p(t, x, y) \leqq c_4 t^{-d_s/2} \exp(-c_5(|x-y|^{d_w}/t)^{1/(d_w-1)}), \quad x, y \in \tilde{F} .$$

Proof. Fix $x \neq y$ and t and let $\varepsilon < \frac{1}{6}|x-y|$, $C_x = B(x, \varepsilon) \cap \tilde{F}$, $C_y = B(y, \varepsilon) \cap \tilde{F}$, $\nu_x = \mu|_{C_x}$, $\nu_y = \mu|_{C_y}$, $A_1 = \{z : |z-x| \leqq \frac{1}{2}|z-y|\} \cap \tilde{F}$, $A_2 = A_1^c \cap \tilde{F}$, and

$$S = \inf\{t : |X_t - X_0| > \tfrac{1}{3}|x-y|\} .$$

Then

$$\begin{aligned} P^{\nu_x}(X_t \in C_y) &= P^{\nu_x}(X_t \in C_y, X_{t/2} \in A_1) + P^{\nu_x}(X_t \in C_y, X_{t/2} \in A_2) \\ &= I_1 + I_2 . \end{aligned}$$

For $z \in C_x$, by Theorem 3.4(b)

$$P^z(X_{t/2} \in A_2) \leqq P^z(S < t/2) \leqq c_6 \exp(-c_7(|z-x|^{d_w}/t)^{1/(d_w-1)}) ,$$

while if $q(z) = P(X_t \in C_y | X_{t/2} = z)$ then by Theorem 6.1(d)

$$q(z) = \int_{C_y} p(t/2, z, w)\mu(dw) \leqq c_1 t^{-d_s/2}\mu(C_y) .$$

Hence

$$I_2 = E^{\nu_x}(q(X_{t/2}); X_{t/2} \in A_2)$$
$$\leqq c_8 \mu(C_x)\mu(C_y) t^{-d_s/2} \exp(-c_5(|z-x|^{d_w}/t)^{1/(d_w-1)}) .$$

To handle I_1, note that by the symmetry of $p(t, x, y)$,

$$P^{\nu_x}(X_t \in C_y, X_{t/2} \in A_1) = P^{\nu_y}(X_t \in C_x, X_{t/2} \in A_1) ,$$

which may be bounded in exactly the same way as I_2.

Adding the bounds for I_1 and I_2,

$$P^{\nu_x}(X_t \in C_y) \leqq c_4 \mu(C_x)\mu(C_y) t^{-d_s/2} \exp(-c_5(|x-y|^{d_w}/t)^{1/(d_w-1)}) .$$

Dividing both sides by $\mu(C_x)\mu(C_y)$, letting $\varepsilon \to 0$, and using the continuity of $p(t, x, y)$ in each variable proves the theorem. □

7 Lower bounds

Lemma 7.1 *There exists $c_1 > 0$ such that*

$$p(t, x, x) \geqq c_1 t^{-d_s/2} .$$

Proof. Recall that by Theorem 3.4(a)

$$P^x(\sigma_r(x) \leqq t) \leqq c_2 \exp(-c_3(t_F^r t)^{-1/(d_w-1)}) .$$

Fix s and pick a so that $c_2 \exp(-c_3 a^{-1/(d_w-1)}) \leqq \frac{1}{2}$. Let $r = [\log(a/s)/\log t_F]$. Then

$$P^x(X_s \in D_r(x)) \geqq P^x(\sigma_r(x) > s) \geqq \tfrac{1}{2} . \tag{7.1}$$

Moreover,

$$\mu(D_r(x)) \leqq 4 m_F^{-r} \leqq c_5 s^{d_s/2} . \tag{7.2}$$

Using Cauchy-Schwarz,

$$\frac{1}{4} \leqq [P^x(X_s \in D_r(x))]^2 = \left(\int_{D_r(x)} p(s, x, y)\mu(dy) \right)^2$$
$$\leqq \mu(D_r(x)) \int_{D_r(x)} p(s, x, y)^2 \mu(dy)$$
$$\leqq \mu(D_r(x)) p(2s, x, x) .$$

Hence, $p(2s, x, x) \geqq (4\mu(D_r(x)))^{-1}$, and this with (7.2) completes the proof. □

Proposition 7.2 *There exist c_{10} and c_{11} such that*

$$p(t, x, y) \geqq c_{11} t^{-d_s/2} \quad \textit{for } |x-y| \leqq c_{10} t^{1/d_w}.$$

Proof. Set $c_{10} = (\frac{1}{2} c_1 c_{6.2}^{-1})^{1/(d_w-d_f)}$. Then $a \leqq c_{10} t^{1/d_w}$ implies that $c_{6.2} t^{-1} a^{d_w-d_f} \leqq \frac{1}{2} c_1 t^{-d_s/2}$. So by Theorem 6.1, if $|x-y| \leqq c_{10} t^{1/d_w}$ then

$$p(t, x, y) \geqq p(t, x, x) - |p(t, x, y) - p(t, x, x)|$$
$$\geqq c_1 t^{-d_s/2} - c_{6.2} t^{-1} |x-y|^{d_w-d_f}$$
$$\geqq \tfrac{1}{2} c_1 t^{-d_s/2} .$$

□

We now use a chaining argument (cf. [FaS]) to obtain the lower bound. First however we need a geometrical result. For $x, y \in \tilde{F}$ let $d(x, y)$ be the length of the shortest path in $\tilde{F}$ connecting x and y.

Lemma 7.3 *There exists a constant c_{17}, depending only on F_1, such that for $x, y \in \tilde{F}$*

$$|x - y| \leqq d(x, y) \leqq c_{17}|x - y| .$$

Proof. For $x \in \tilde{F}$ let $\phi_n(x)$ denote the lower left hand corner of one of the squares S in $\mathscr{S}_n$ containing x, where we adopt some procedure for breaking ties. Set

$$H_n = \bigcup\{\partial S : S \in \mathscr{S}_n, S \subset \tilde{F}\} ,$$

and note that (2.1)(H4) implies that $H_n \subset \tilde{F}$. For $x, y \in H_n$ write $d_n(x, y)$ for the length of the shortest path in H_n connecting x and y. Let

$$c_{16} = \sup\{d_1(0, y) : y \in F \cap H_1\} .$$

Let $x \in F$. By the scaling symmetry of F it is clear that $d_n(\phi_{n+1}(x), \phi_n(x)) \leqq l^{-n}c_{16}$. Hence

$$d(x, 0) \leqq \sum_{n=0}^{\infty} d_n(\phi_{n+1}(x), \phi_n(x)) \leqq 2c_{16}.$$

Similarly we have

$$d(\phi_n(x), x) \leqq 2c_{16}l^{-n} . \tag{7.4}$$

Now let $x, y \in \tilde{F}$, and choose m such that $y \in D_m(x) - D_{m+1}(x)$. Then $|x - y| \geqq cl^{-m}$. Let z be the center of $D_m(x)$, so that z is one of the corners of a square $S \in \mathscr{S}_{m-1}$ containing x. By (7.4) we have $d(x, y) \leqq d(x, z) + d(z, y) \leqq 4c_{16}l^{-m}$, and after rearranging, the result follows. □

Theorem 7.4 *There exist c_{18}, c_{19} such that*

$$p(t, x, y) \geqq c_{18}t^{-d_s/2}\exp(-c_{19}(|x - y|^{d_w}/t)^{1/(d_w-1)}), \quad x, y \in \tilde{F}.$$

Proof. Write $D = c_{17}|x - y|$. The theorem is immediate for the case $D \leqq c_{17}c_{10}t^{1/d_w}$ by Proposition 7.2, so suppose that $D > c_{20}t^{1/d_w}$, where $c_{10} = c_{10}c_{17}$.

We may find c_{21} depending only on c_{20} and d_w such that if we let n be the largest integer less than or equal to $c_{21}t^{-1/(d_w-1)}D^{d_w/(d_w-1)}$, then $n \geqq 4$ and $3D/n \leqq c_{10}(t/n)^{1/d_w}$. Let $x_0 = x, x_n = y$, and pick $x_1, x_2, \ldots, x_{n-1} \in \tilde{F}$ such that $d(x_{i+1}, x_i) \leqq 2D/n$. Let $\varepsilon = D/n$ and $B_i = B(x_i, \varepsilon) \cap \tilde{F}$. Note that if $z \in B_i$,

$$|x_{i-1} - z| \leqq 2D/n + \varepsilon \leqq 3D/n \leqq c_{10}(t/n)^{1/d_w},$$

so that $p(t/n, x_{i-1}, z) \geqq c_{11}(t/n)^{-d_s/2}$. Then

$$\begin{aligned} p(t, x, y) &\geqq \\ &\int_{B_1} \cdots \int_{B_{n-1}} p\left(\frac{t}{n}, x, y_1\right) \ldots p\left(\frac{t}{n}, y_{n-2}, y_{n-1}\right)p\left(\frac{t}{n}, y_{n-1}, y\right)\mu(dy_1) \ldots \mu(dy_{n-1}) \\ &\geqq \left(\prod_{i=1}^{n-1} \mu(B_i)\right)c_{11}^n(t/n)^{-d_s n/2} \\ &\geqq c_{19}^n(D/n)^{d_f(n-1)}(t/n)^{-d_s n/2}. \end{aligned}$$

Since $d_s/2 = d_f/d_w$ and by our choice of n, $(D/n)/(t/n)^{1/d_w}$ is bounded above and below by positive constants, independent of D and t,

$$\begin{aligned} p(t, x, y) &\geqq c_{21}^n c_{22}(t/n)^{-d_f/d_w} \\ &\geqq c_{21}^n c_{23} t^{-d_s/2} \\ &= c_{23} t^{-d_s/2} \exp(-n \log c_{21}^{-1}) , \end{aligned} \tag{7.5}$$

where $c_{21} < 1$. Substituting our choice of n in (7.5) completes the proof. □

Combining Theorems 6.1, 6.2 and 7.4 we have Theorem 1.1.

8 Further results

1 *Properties of the process X*

Given Theorem 1.1 and the various estimates used in its proof, we can derive a number of properties of the process X. As the proofs are essentially the same as those in [BP] for the Sierpinski gasket, we just state the results.

Theorem 8.1 (a) *There are constants c_1, c_2 such that*

$$c_1 t^{p/d_w} \leqq E^x |X_t - x|^p \leqq c_2 t^{p/d_w} .$$

(b) *X has a modulus of continuity given by*

$$c_3 \leqq \lim_{\delta \downarrow 0} \sup_{\substack{0 \leqq s \leqq t \leqq T \\ |s-t| \leqq \delta}} \frac{|X_t - X_s|}{|s-t|^{1/d_w} (\log 1/|s-t|)^{(d_w - 1)/d_w}} \leqq c_4 .$$

(c) *If $T_x^+ = \inf\{t > 0 : X_t = x\}$ then $P^x(T_x^+ = 0) = 1$, so that for all $x \in \tilde{F}$, x is regular for $\{x\}$.*
(d) *For each $x, y \in \tilde{F}$, $P^x(T_y < \infty) = 1$.*
(e) *$\{t : X_t = x\}$ is P^y-a.s. perfect and unbounded, so that X is point recurrent.*

2 *Local time*

While it was proved in [BB2] that the local time L_t^x of X exists and is jointly continuous, we did not obtain the best modulus of continuity. Applying the techniques of [MR] and [B] we have

Theorem 8.2 *There exists a jointly continuous version L_t^x of the local time of X which satisfies the density of occupation formula* (3.3) *and has modulus of continuity given by*:

$$\lim_{\delta \downarrow 0} \sup_{\substack{0 \leqq s \leqq t \\ |x-y| \leqq \delta}} \frac{|L_s^x - L_s^y|}{\varphi(|x-y|)} \leqq c_5 \left(\sup_{z \in \tilde{F}} L_t^z \right)^{1/2} , \tag{8.1}$$

where $\varphi(u) = u^{\frac{1}{2}(d_w - d_f)} (\log 1/u)^{1/2}$.

3 Infinitesimal generator of X

Let $C_0(\tilde{F})$ be the set of continuous functions on $\tilde{F}$ vanishing at ∞. For $f \in C_0(\tilde{F})$ set

$$P_t f(x) = E^x f(X_t);$$

the estimate (1.1) shows that $P_t: C_0(\tilde{F}) \to C_0(\tilde{F})$, and by Theorem 1.1(e) (P_t) is a strong Feller semigroup on $C_0(\tilde{F})$. Let $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$ be the infinitesimal generator of (P_t). We call $\mathscr{L}$ a *Laplacian* on $\tilde{F}$.

Theorem 8.3 *Every function in $\mathscr{D}(\mathscr{L})$ is Hölder continuous of order $d_w - d_f$.*

Proof. This is immediate from Theorem 4.9(b) and the fact that $\mathscr{D}(\mathscr{L}) = \{U^\lambda f : f \in C_0(\tilde{F})\}$. □

The following result is proved in just the same way as [BP, Theorem 7.10]:

Theorem 8.4 *For $y_0 \in \tilde{F}$ the function $p(\cdot, \cdot, y_0)$ is a solution of the heat equation on $\tilde{F}$:*

$$\frac{\partial}{\partial t} p(t, x, y_0) = \mathscr{L}_x p(t, x, y_0), \quad t > 0, x \in \tilde{F}.$$

4 Absorbing Brownian motion and spectral dimension

Let X^F be the process X_t killed on exiting F and let $\bar{p}(t, x, y)$ be the corresponding transition density. Just as in [BP, Theorem 7.11], $\bar{p}(t, x, y)$ is jointly continuous on $(0, \infty) \times F \times F$, $\bar{p}(t, x, y) \leqq p(t, x, y)$, and if $\delta \in (0, 1)$, $t_0 > 0$, then

(8.2) $$\bar{p}(t, x, y) \geqq c_6(\delta) t^{-d_s/2} \exp(-c_7(\delta)(|x - y|^{d_w}/t)^{1/(d_w - 1)})$$

for $t \in [0, t_0]$, $x, y \in F \cap [0, \delta)^2$.

For large t, $\bar{p}(t, x, y)$ tends to 0 exponentially fast. This is clear from the eigenvalue expansion (5.3).

As X is continuous the Laplacian $\mathscr{L}$ is a local operator, and the infinitesimal generator $(\mathscr{L}_F, \mathscr{D}(\mathscr{L}_F))$ of the semigroup of X^F is just $\mathscr{L}$ acting on the domain $\mathscr{D}(\mathscr{L}_F) = \{f \in \mathscr{D}(\mathscr{L}) : f = 0 \text{ on } F^c \cap \tilde{F}\}$. Thus the λ_j in (5.3) are the eigenvalues of $-\mathscr{L}_F$.

Set $N(\lambda) = \#\{\lambda_j : \lambda_j \leqq \lambda\}$. The *spectral dimension* of $\tilde{F}$ is defined to be

(8.3) $$\lim_{\lambda \to \infty} \frac{\log N(\lambda)}{\log \lambda}$$

if this limit exists. (See [RT], [W]). Using the estimate (8.2) it follows, just as in [BP, pp. 618–619] that

(8.4) $$c_8 \lambda^{d_s/2} \leqq N(\lambda) \leqq c_9 \lambda^{d_s/2}, \quad \lambda \geqq \lambda_1.$$

Thus the spectral dimension of F does exist, and equals the number d_s defined in (2.4). It seems very likely that, as in the case of the Sierpinski gasket (see [FS]),

(8.5) $$\liminf_{\lambda \to \infty} \lambda^{-d_s/2} N(\lambda) < \limsup_{\lambda \to \infty} \lambda^{-d_s/2} N(\lambda).$$

5 *Reflecting Brownian motion*

Suppose we change $\partial_a F$ to a reflecting boundary and let $p^R(t, x, y)$ be the corresponding transition densities Clearly $p^R(t, x, y) \geqq \bar{p}(t, x, y)$ so we get a lower bound on $p^R(t, x, y)$ for $x, y \in F \cap [0, \delta)^2, t \leqq t_0$.

It is not hard to show, using an argument similar to that of Theorem 6.2, that

$$p^R(t, x, y) \leqq c_{10}(\delta) t^{-d_s/2} \exp(-c_{11}(\delta)(|x - y|^{d_w}/t)^{1/(d_w - 1)}),$$

for $0 < t \leqq t_0$, $x, y \in F \cap [0, \delta)^2$.

As $t \to \infty$, $p^R(t, x, y) \to \mu(F)^{-1} = 1$ uniformly and exponentially fast. This may be proved by using an eigenvalue expansion for p^R; cf. [BH], Theorem 2.4.

6 *Einstein relation*

Recall from (2.2) that

$$\tfrac{1}{4} \rho_F^n \leqq R_n \leqq 4 \rho_F^n .$$

The mathematical physics literature calls

$$\tilde{\zeta} = \log \rho_F / \log l$$

the resistance exponent of the Sierpinski carpet. From our definitions of d_s and d_w, it is immediate that

$$2 d_f / d_s = d_f + \tilde{\zeta}, \tag{8.6}$$

which is known as the Einstein relation.

7 *Sierpinski carpets not satisfying (H4)*

The only place in this paper where the hypothesis (2.1)(H4) is used is in Lemma 7.3, where it ensures that the intrinsic distance $d(x, y)$ is comparable with the Euclidean distance $|x - y|$. We now summarize briefly how the results of this paper have to be modified in the case when (H4) does not hold.

First, some more work on the geometry of $\tilde{F}$ is necessary – see [BS] for similar results on nested fractals. Let b_n be the smallest number of squares in $\mathscr{S}_n(F_n)$ required to form a strip connecting two opposite sides of F_n. (It is clear that $b_n \geqq l^n$, and that if (H4) holds then $b_n = l^n$). Let $d_n(x, y)$, $x, y \in F_n$, be the length of the shortest path in F_n connecting x and y. The (b_n) satisfy

$$\tfrac{1}{8} b_n b_m \leqq b_{n+m} \leqq b_n b_m ,$$

and so, as in [BB3, Theorem 5.1] there exists a constant $b_F \geqq l$ such that $b_F^n \leqq b_n \leqq 8 b_F^n$ for all $n \geqq 0$. Define the *chemical exponent* of $\tilde{F}$ by

$$d_c = \log b_F / \log l . \tag{8.7}$$

One then has

$$d_n(x, y) \approx |x - y|^{d_c} (b_F / l)^n , \tag{8.8}$$

and taking limits, possibly along a subsequence, one obtains a metric $d(x, y)$ on $\tilde{F}$ (the 'chemical distance' – see [HBA]) which satisfies

(8.9) $$d(x, y) \approx |x - y|^{d_c}, \quad x, y \in \tilde{F} .$$

The upper bound in Proposition 3.3 is still valid if $d_c > 1$, but is no longer the best possible result. One can take $m_r = \frac{1}{2} b_r - 2$ in the proof, which leads to the bound

$$P^x(\sigma_0(x) \leqq s) \leqq c_{12} \exp(-c_{13} s^{-d_c/(d_w - d_c)}) .$$

The proof of the upper bound on $p(t, x, y)$ then proceeds exactly as before, but with an exponent of $d_c/(d_w - d_c)$ instead of $1/(d_w - 1)$ in the exponential, to give

(8.10) $$p(t, x, y) \leqq c_{14} t^{-d_f/d_w} \exp(-c_{15}(|x - y|^{d_w}/t)^{d_c/(d_w - d_c)}), \quad x, y \in \tilde{F}.$$

For the lower bound the chaining argument Theorem 7.4 requires only minor modification. Two points x and y in $\tilde{F}$ with $|x - y| = D$ are connected by a strip of at most $c b_F^n D^{d_c}$ squares in $\tilde{F}_n$. Thus $c D^{d_c} \varepsilon^{-d_c}$ balls of Euclidean radius ε are required to link x and y. Choosing $\varepsilon = (t/n)^{1/d_w}$, and $n = c D^{d_c} \varepsilon^{-d_c}$ gives the lower bound

(8.11) $$p(t, x, y) \geqq c_{16} t^{-d_f/d_w} \exp(-c_{17}(|x - y|^{d_w}/t)^{d_c/(d_w - d_c)}), \quad x, y \in \tilde{F} .$$

While at first sight (8.10) and (8.11) appear to be a considerable generalisation of (1.1), if these bounds are rewritten using the chemical distance $d(x, y)$, then they assume very much the same form. Set

(8.12) $$d_f^l = \log m_F / \log b_F, \quad d_w^l = \log t_F / \log b_F ;$$

then d_f^l is the Hausdorff dimension of $\tilde{F}$ with respect to the chemical metric d. Since $d_f^l = d_f/d_c$, and $d_w^l = d_w/d_c$, using (8.8) one obtains

(8.13) $$p(t, x, y) \leqq c_{18} t^{-d_f^l/d_w^l} \exp(-c_{19}(d(x, y)^{d_w^l}/t)^{1/(d_w^l - 1)}), \quad x, y \in \tilde{F} ,$$

and a corresponding lower bound.

We remark that Kumagai and Hambly have obtained similar estimates for the transition density of Brownian motion on nested fractals.

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