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SHORT COMMUNICATION

HITTING TIME OF A MOVING BOUNDARY FOR A DIFFUSION

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We obtain asymptotic estimates for the quantity $r=\log \mathbf{P}[T_f>t]$ as $t\to\infty$ where $T_f=\inf\{s: |X(s)|>f(s)\}$ and X is a real diffusion in natural scale with generator $a(x)\,\mathrm{d}^2(\cdot)/\mathrm{d}x^2$ and the 'boundary' $f(s)$ is an increasing function. We impose regular variation on a and f and the result is expressed as $r=-\int_0^t \lambda_1(f(s))\,\mathrm{d}s(1+\mathrm{o}(1))$ where $\lambda_1(f)$ is the smallest eigenvalue for the process killed at $\pm f$.

Diffusion	eigenvalue
regular variation	moving boundary

1. Introduction

Consider a diffusion $X=\{X(t), t\geq 0\}$ on the line with differential generator of the form

$$Au(x)=a(x)u''(x), \quad -\infty<x<\infty,$$

where $a(x)$ is a non-negative function such that $1/a$ is locally integrable. Let $f(t)$ be a smooth increasing deterministic non-negative function and define

$$T_f=\inf\{t:|X(t)|>f(t)\}.$$

We call T_f the *hitting time* of the (symmetric) moving boundary $\pm f$. In this paper we show that, under certain regularity conditions on a and f,

$$\log \mathbf{P}[T_f>t]\sim -\int_0^t \lambda_1(f(s))\,\mathrm{d}s, \quad t\to\infty, \tag{1.1}$$

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where the function λ_1 is the smallest eigenvalue of the boundary value problem

$$Au(x)+\lambda_1 u(x)=0, \quad -l<x<l, \qquad u(-l)=u(l)=0.$$

(We think of $\lambda_1=\lambda_1(l)$ as a function of the interval $[-l, l]$, $l=f(t)$.)

Unfortunately our regularity conditions are rather stringent (see Theorem 3.1). Nevertheless, as we are sure the reader will agree, the simple asymptotic formula (1.1) ought to be the rule in moving boundary problems, and we challenge the reader to find the most general conditions for its validity. Note that (1.1) always holds when f is a constant (see the next section).

2. Estimates for horizontal boundary

In this section we obtain estimates for the probabilities

$$q(t, z, l)\equiv P_z[|X_s|\leq l \text{ for all } 0\leq s\leq t] \tag{2.1}$$

using the eigenfunction expansion for the transition densities of the process $\tilde{X}$: X killed at $\pm l$.

Let $a(x)$ be a non-negative Borel function on R^1 with only isolated discontinuities and such that

$$0<\int_{l_1}^{l_2} a(x)^{-1}\,dx<\infty \quad \text{for all } -\infty<l_1<l_2<\infty. \tag{2.2}$$

(The continuity assumption can be weakened but (2.2) is essential.) Let X be the diffusion on R^1 with generator

(i) $\quad Au(x)=a(x)u''(x), \quad -\infty<x<\infty,$

acting on the domain (2.3)

(ii) $\quad \mathscr{D}=\{u: u, u', au'' \text{ are bounded and continuous on } R^1\}.$

Under our assumptions we have that

(i) $\quad P^x[T_y<\infty]=1 \quad$ for all x, y,

(ii) $\quad 0<E^x T_a\wedge T_b<\infty \quad$ for $-\infty<a<x<b<\infty$. (2.4)

where

$$T_a=\min\{t: X_t=a\}$$

(see [3, Chapter 4]). Fix $l>0$ and let $\tilde{X}$ be the process X killed at $\pm l$:

$$\tilde{X}_t=\begin{cases} X_t & \text{for } 0\leq t<\sigma_l\equiv T_l\wedge T_{-l},\\ \infty & \text{for } t>\sigma_l.\end{cases}$$

Then $\tilde{X}$ has generator $\tilde{A} = a\ \mathrm{d}^2/\mathrm{d}x^2$, acting on

$$\tilde{\mathscr{D}} = \{u : u(l-) = u(-l+) = 0,\ u, u', au'' \in bC(l-, l)\}.$$

$\tilde{\mathscr{D}}$ is dense in the Hilbert space $\mathscr{H} = \mathscr{L}^2\{(-l, l), a(x)^{-1}\,\mathrm{d}x\}$ and $\tilde{A}$ is self-adjoint on $\tilde{\mathscr{D}} \cap \mathscr{H}$ with a negative discrete spectrum $\{-\lambda_n = -\lambda_n(l), n \geq 1\}$ where

$$0 < \lambda_1 < \lambda_2 < \cdots \uparrow \infty.$$

Let $\Psi_n(x) = \Psi_n(x, l)$ be the normalized eigenfunction corresponding to λ_n, i.e.,

$$\begin{aligned} &\text{(i)} \quad a(x)\Psi_n''(x) = -\lambda_n \Psi_n(x), \quad -l < x < l, \qquad \Psi_n(l) = \Psi_n(-l) = 0, \\ &\text{(ii)} \quad \int_{-l}^{l} \Psi_n(x)^2 a(x)^{-1}\,\mathrm{d}x = 1. \end{aligned} \tag{2.5}$$

It may be shown that Ψ_1 has no zero in $(-l, l)$ so we may also suppose that

$$\Psi_1(x, l) > 0, \quad -l < x < l.$$

Lemma 2.1. *Put*

$$a_l(x) = a(xl)/a(l), \quad -1 \leq x \leq 1,$$

$$\bar{a}(x) = \sup_{l \geq l_0} a_l(x), \qquad \underline{a}(x) = \inf_{l \geq l_0} a_l(x).$$

If for some l_0

$$\bar{a}(x) < \infty, \quad 0 < |x| \leq 1 \ (\bar{a}(0) = \infty \textit{ allowed}), \tag{2.6}$$

$$\int_{-1}^{1} \underline{a}(x)^{-1}\,\mathrm{d}x < \infty, \tag{2.7}$$

then there are positive constants C_1, C_2 *so that for all* $l \geq l_0$

$$C_1 n^2 \leq \lambda_n(l) l^2/a(l) \leq C_2 n^2, \quad n = 1, 2, \ldots. \tag{2.8}$$

If, in addition to (2.6) *and* (2.7), *we have*

$$a_l(x) \to |x|^\alpha \quad \text{as } l \to \infty, -\infty < x < \infty, \tag{2.9}$$

where

$$-\infty < \alpha < 1, \tag{2.10}$$

then, as $l \to \infty$,

$$\lambda_n(l) l^2/a(l) \to \mu_n, \quad n = 1, 2, \ldots, \tag{2.11}$$

and there are constants C_3, l_1 *so that for* $l \geq l_1$

$$C_3 n^2 \leq [\lambda_n(l) - \lambda_1(l)] l^2/a(l), \quad n = 2, 3, \ldots. \tag{2.12}$$

In (2.11) *the* μ_n *are the eigenvalues of the boundary value problem*

$$|x|^\alpha y''(x) = -\mu y(x), \quad -1 < x < 1, \qquad y(1) = y(-1) = 0. \tag{2.13}$$

Proof. Consider the boundary value problem

$$b(x)y''(x) = -\mu y(x), \quad -1 < x < 1, \qquad y(1) = y(-1) = 0 \tag{2.14}$$

where $b(x)$ is a non-negative Borel function such that

$$0 < \int_c^d b(x)^{-1}\,dx < \infty \quad \text{for all } -1 \leq c < d \leq 1. \tag{2.15}$$

Let $\mu_n\{b\}$ denote the nth eigenvalue. The following three facts are well known.

(1) As functions of b the μ_n are monotone, i.e., if $b_1(x) \leq b_2(x)$ for all $|x| \leq 1$, then $\mu_n\{b_1\} \leq \mu_n\{b_2\}$ for all $n \geq 1$ (see, for example, [4, p. 5.11]).

(2) For a fixed b we have, as $n \to \infty$,

$$\mu_n\{b\} \sim \pi^2 n^2 \left(\int_{-1}^{1} b(x)^{-1/2}\,dx \right)^{-2} \tag{2.16}$$

($\sim$ means the ratio of both sides goes to 1) (see [5]).

(3) With respect to the metric

$$d\{b_1, b_2\} = \int_{-1}^{1} |b_1(x)^{-1} - b_2(x)^{-1}|\,dx, \tag{2.17}$$

the eigenvalues are continuous (see [4, p. 5.11]).

Changing the independent variable $x \to xl$ and dividing by $a(l)$ converts the boundary value problem (2.5) into the boundary value problem (2.14) with $b = a_l$. We immediately see that

$$\mu_n\{a_l\} = \lambda_n(l) l^2 / a(l). \tag{2.18}$$

From (1), (2.6), (2.7) and (2.18) we obtain

$$0 < \mu_n\{\underline{a}\} \leq \lambda_n(l) l^2 / a(l) \leq \mu_n\{\bar{a}\} < \infty \tag{2.19}$$

for all $n \geq 1$. The extreme members, $\mu_n\{\underline{a}\}$ and $\mu_n\{\bar{a}\}$, of (2.19) do not depend on l and (2.8) quickly follows from an application of (2) to them.

From (2.9) and (2.7) and dominated convergence we see that $a_l(x) \to |x|^\alpha$, $l \to \infty$, not only pointwise but also in the metric (2.17). This gives (2.11). This also gives (2.12) since $\mu_n - \mu_1 > 0$ for $n \geq 2$, $(\mu_1 < \mu_2 < \cdots)$, and since $\mu_n \sim n^2 \pi^2 (2-\alpha)^2/16$ by (2.16).

Lemma 2.2. *Let* $\psi_1, \psi_2, \ldots,$ *be as in* (2.5) *and put*

$$m(l) = \int_{-l}^{l} a(x)^{-1}\,dx.$$

Under the same assumptions as in Lemma 2.1 *we can find constants* C_4, l_2 *so that for all* $n = 2, 3, 4, \ldots,$ *and for all* $l \geq l_2$ *we have*

$$|\psi_n(x, l)| / \psi_1(x, l) \leq 2 l m(l) \lambda_n(l) < C_4 n^2 \tag{2.20}$$

and

$$\tfrac{1}{2}m(l)^{-1/2}(l-|x|)/l \leq \psi_1(x) \leq \lambda_1(l)m(l)^{1/2}(l-|x|). \tag{2.21}$$

Proof. ψ_1 is positive, concave and attains its maximum value $\|\psi_1\|_\infty = \psi_1(x_0)$ at a unique point x_0 in $(-l, l)$ with $\psi_1'(x_0)=0$. It follows that for $-l \leq x \leq x_0$

$$\psi_1(x) \geq (l+x_0)^{-1}\psi_1(x_0)(l+x) \geq (\tfrac{1}{2}l)\psi_1(x_0)(l+x)$$

and for $x_0 \leq x \leq l$

$$\psi_1(x) \geq (l-x_0)^{-1}\psi_1(x_0)(l-x) \geq (\tfrac{1}{2}l)\psi_1(x_0)(l-x),$$

or, regardless of the sign of x_0,

$$\psi_1(x) \geq (\tfrac{1}{2}l)\psi_1(x_0)(l-|x|), \quad -l \leq x \leq l.$$

This gives the left-hand side of (2.21) since

$$1 = \int_{-l}^{l} \psi_1(x)^2 a(x)^{-1}\,\mathrm{d}x \leq \psi_1(x_0)^2 m(l).$$

Let r be a zero of ψ_n', $-l<r<l$. Then

$$\psi_n'(x) = \int_r^x \psi_n''(y)\,\mathrm{d}y = -\lambda_n \int_r^x \psi_n(y)a(y)^{-1}\,\mathrm{d}y.$$

Hence, for $-l \leq x < 0$,

$$|\psi_n(x)| = \left|\int_{-l}^{x} \psi_n'(s)\,\mathrm{d}s\right| \leq \lambda_n(l+x)m(l)^{1/2}$$

by Cauchy–Schwarz, and $|\psi_n(x)| \leq \lambda_n(l-x)m(l)^{1/2}$ for $0 \leq x \leq l$. Setting $n=1$ we get the right-hand side of (2.21). Dividing by $\psi_1(x)$ and applying the left-hand side of (2.21) we get the first inequality in (2.20). The last inequality in (2.20) results from (2.8) and the asymptotic formula

$$m(l) \sim 2(1-\alpha)^{-1} l a(l)^{-1} m, \quad l \to \infty, \tag{2.22}$$

a standard result from the theory of regular variation (see [1, pp. 275–284]).

Remark 2.3. The only point at which we used the assumptions (2.6), (2.7) and (2.9) is to get the last inequality in (2.20); the other inequalities in Lemma 2.2 are therefore valid without these assumptions.

We now come to the main result of this section.

Lemma 2.4. *Assume* (2.6), (2.7) *and* (2.9) *with* $-\infty<\alpha<1$. *For any* $\varepsilon>0$, $0<\varepsilon<1$, *there is a* $\delta>0$, *depending only on* ε, *so that if*

$$ta(l)/l^2 \geq \delta \quad \text{and} \quad l \geq \max\{l_0, l_1, l_2\} = \mathscr{L}_0 \tag{2.23}$$

(l_0, l_1, l_2 as in Lemmas 2.1 *and* 2.2), *then*

$$q(t, x, l) \lesseqgtr (1 \pm \varepsilon) m(l)^{1/2} \psi_1(x; l)\, e^{-\lambda_1(l)t} \tag{2.24}$$

where $q(t, x, l)$ is as defined in (2.1).

Proof. Let $\tilde{p}_t(x, y)$ be the transition density of $\tilde{X}$ with respect to its speed measure $a(y)^{-1}\,dy$. Then (see [3, pp. 149–154])

$$\tilde{p}_t(x, y) = \sum_{m=1}^{\infty} e^{-\lambda_n t} \psi_n(x) \psi_n(y)$$

where λ_n, ψ_n are as in (2.4)–(2.5). Now

$$\int_{-l}^{l} |\psi_n(y)| a(y)^{-1}\, dy \leqslant m(l)^{1/2}$$

by Cauchy–Schwarz, so

$$q(t, x, l) = \int_{-l}^{l} \tilde{p}_t(x, y) a(y)^{-1}\, dy \lesseqgtr m(l)^{1/2} \Big(e^{-\lambda_1 t} \psi_1(x) \pm \sum_{n=2}^{\infty} e^{-\lambda_n t} |\psi_n(x)| \Big). \tag{2.25}$$

Assume t and l satisfy (2.23). Then

$$e^{-(\lambda_n - \lambda_1)t} \leqslant e^{-C_3 n^2 t a(l)/l^2} \leqslant e^{-C_3 \delta n^2}$$

by (2.12), and

$$|\psi_n(x)| \leqslant C_4 n^2 \psi_1(x)$$

by (2.20). Hence

$$\sum_{n=2}^{\infty} e^{-\lambda_n t} |\psi_n(x)| \leqslant C_4 \psi_1(x)\, e^{-\lambda_1 t} \sum_{n=2}^{\infty} n^2 e^{-C_3 \delta n^2}. \tag{2.26}$$

For $n \geqslant 2$, $n^2 e^{-C_3 \delta n^2}$ is a decreasing function of n provided

$$\delta \geqslant (4C_3)^{-1}. \tag{2.27}$$

In that case we get

$$\sum_{n=2}^{\infty} n^2 e^{-C_3 \delta n^2} < \int_1^{\infty} x^2 e^{-C_3 \delta x^2}\, dx = \tfrac{1}{2}(C_3\delta)^{-3/2} \int_{C_3 \delta}^{\infty} y^{-1/2} e^{-y}\, dy$$
$$< \tfrac{1}{2}(C_3 \delta)^{-2} e^{-C_3 \delta} \quad \text{(integrate by parts).} \tag{2.28}$$

Combining (2.25), (2.26) and (2.28) we get

$$q(t, x, l) \lesseqgtr m(l)^{1/2} \psi_1(x)\, e^{-\lambda_1 t} (1 \pm \varepsilon(\delta))$$

where

$$\varepsilon(\delta) = \tfrac{1}{2} C_4 C_3^{-2} \delta^{-2} e^{-C_3 \delta}$$

whenever $\delta \geqslant (4C_3)^{-1}$. This yields (2.24) since $\varepsilon(\delta) \to 0$ as $\delta \to \infty$.

Corollary 2.5. *There exist constants K_1, K_2, δ_0 so that if θ is any number, $0<\theta<1$, then*

$$K_1(1-\theta)\,\mathrm{e}^{-\lambda_1(l)t} < q_t(x,l) < K_2(1-\theta)\,\mathrm{e}^{-\lambda_1(l)t} \tag{2.29}$$

whenever

$$|x|\leqslant \theta l, \qquad l\geqslant \mathscr{L}_0, \qquad ta(l)/l^2\geqslant \delta_0. \tag{2.30}$$

Proof. This easily follows from (2.8), (2.21), (2.22) and (2.24).

Remark 2.6. One should note that the existence of the limit of $a(lx)/a(l)$ as $l\to\infty$ for all x forces the limit to be as in (2.9), namely $|x|^\alpha$. Further, (2.9) forces an asymptotic symmetry: $a(x)/a(-x)\to 1$, $x\to\infty$.

Remark 2.7. The eigenfunction expansion for $\tilde{p}$ is valid under (2.2) alone. Hence

$$\log \mathbf{P}[\sigma_l > t] \sim -\lambda_1(l)t, \quad t\to\infty,$$

for l fixed. This is the same as (1.1) when $f\equiv l$.

3. Estimates for curved boundary

In this section we study the asymptotic behavior of the probabilities

$$\mathbf{P}[T_f > t] = \mathbf{P}[|X(s)|\leqslant f(s) \text{ for } 0\leqslant s\leqslant t]$$

as $t\to\infty$ where

$$T_f = \inf\{s: |X(s)| > f(s)\}$$

and X is the diffusion on R' with generator (2.3).

Throughout this section we assume that $a(x)$ satisfies the assumptions of Lemma 2.4 and we assume that the function f satisfies

$$\begin{aligned} &\text{(i)} \quad f'(t)>0, f''(t)<0 \quad \text{for all } t\geqslant 0,\\ &\text{(ii)} \quad f(0)>0 \quad \text{and} \quad f(\infty-)=\infty. \end{aligned} \tag{3.1}$$

We will impose additional restrictions on f below. Fix $\varepsilon>0$ and choose numbers $t_0<t_1<\cdots$ and $l_1<l_2<\cdots$ to satisfy $f(t_0)=l_1$ and

$$f(t_j) = l_{j+1} = (1+\varepsilon)l_j, \quad j=1,2,\ldots.$$

The t_j exist on account of (3.1). Put

$$I_j = (t_{j-1}, t_j], \qquad \Delta_j = |I_j| = t_j - t_{j-1},$$

and note that

$$\Delta_1 < \Delta_2 < \cdots < \Delta_j \uparrow \infty \quad \text{as } j \to \infty.$$

Define events A_j, B_j by

$$A_j = \{|X(s)| \leq l_j \text{ for all } s \in I_j\}, \qquad B_j = \{|X(s)| \leq l_{j+1} \text{ for all } s \in I_j\}.$$

Let us write $\mathbf{P}[\cdot]$ for $P_0[\cdot] = \mathbf{P}[\cdot \mid X(0) = 0]$. Clearly

$$\mathbf{P}[A_1 A_2 \cdots A_{n+1}] \leq \mathbf{P}[T_f > t] \leq \mathbf{P}[B_1 B_2 \cdots B_n] \tag{3.2}$$

whenever $n = n(t)$ satisfies

$$t_n \leq t < t_{n+1}.$$

Since X is homogeneous Markov, we have

$$\mathbf{P}[B_1 B_2 \cdots B_n] = \mathbf{E}\{P_{X(t_{n-1})}(|X(t)| \leq l_{n+1},\, t \in (0, \Delta_n]),\, B_1 B_2 \cdots B_{n-1}\}. \tag{3.3}$$

On B_{n-1}, $|X(t_{n-1})| \leq l_n = (1+\varepsilon)^{-1} l_{n+1}$, consequently, by Corollary 2.5,

$$P_{X(t_{n-1})}(|X(t)| \leq l_{n+1},\, t \in (0, \Delta_n]) \leq \varepsilon k_2 (1+\varepsilon)^{-1}\, \mathrm{e}^{-\lambda_1(l_{n+1})\Delta_n}, \tag{3.4}$$

provided the assumptions of Corollary 2.5 hold. The constant k_2 only depends on δ_0, L_0 and not on ε, l_{n+1}, Δ_{n+1}. Let us now assume

$$\lim_{\varepsilon \downarrow 0} \liminf_{n \to \infty} \Delta_n a(l_n)/l_n^2 = \infty. \tag{3.5}$$

Later we will impose conditions on f which force (3.5). For all sufficiently small $\epsilon > 0$ it follows from (3.5) that for some j_0, depending on ε and δ_0 as in (2.30),

$$\Delta_j a(l_{j+1})/l_{j+1}^2 \geq \delta_0 \quad \text{for all } j \geq j_0, \tag{3.6}$$

and then from (3.3), (3.4), for all $n \geq j_0$,

$$\mathbf{P}[B_1 \cdots B_n] \leq (\varepsilon k_2 (1+\varepsilon)^{-1})^{n-j_0+1} \exp\left\{-\sum_{j=j_0}^{n} \lambda_1(l_{j+1})\Delta_j\right\} \mathbf{P}[B_1 \cdots B_{j_0-1}]. \tag{3.7}$$

Suppose that

$$\varepsilon k_2 \leq 1.$$

Simplifying (3.7) and combining with the right-hand side of (3.2) gives

$$\log \mathbf{P}[T_f > t] \leq -\sum_{j=j_0}^{n} \lambda_1((1+\varepsilon) f(t_{j-1}))\Delta_j \leq -\int_{t_{j(0)}}^{t_n} \lambda_1((1+\varepsilon) f(s))\, \mathrm{d}s. \tag{3.8}$$

(Note that $s \mapsto \lambda_1((1+\varepsilon)f(s))$ is decreasing by (3.1) and (2) from Section 2.) By a very similar calculation, using the left-hand side of (3.2) and (2.29), we get

$$\log \mathbf{P}[T_f > t] \geq c_1 - n c_2 - \int_{t_{j(0)}}^{t_{n+1}} \lambda_1((1+\varepsilon)^{-1} f(s))\, \mathrm{d}s \tag{3.9}$$

where c_1, c_2 do not depend on t or n. In fact

$$c_2 = |\log(\varepsilon k_1/(1+\varepsilon))|, \qquad c_1 = \log \mathbf{P}[B_1 \cdots B_{j-1}] - j_0 c_2.$$

If we now also assume, in addition to (3.1) and (3.5),

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \int_{t_n}^{t_{n+1}} \lambda_1(f(s))\,\mathrm{d}s \Big/ \int_0^{t_n} \lambda_1(f(s))\,\mathrm{d}s = 0, \tag{3.10}$$

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} nc_2 \Big/ \int_0^{t_n} \lambda_1(f(s))\,\mathrm{d}s = 0, \tag{3.11}$$

then from (3.8)–(3.9) and regular variation of λ_1 ($\lambda_1(\theta f) \sim \theta^{\alpha-2}\lambda_1(f), f\to\infty$, by (2.9)–(2.11)), we get that $T_f < \infty$ a.s. and that

$$\log \mathbf{P}[T_f > t] \sim -\int_0^t \lambda_1(f(s))\,\mathrm{d}s, \quad t\to\infty. \tag{3.12}$$

(As usual, $\sim$ means the ratio of both sides goes to 1.) We now come to the main result of this paper.

Theorem 3.1. *Let a satisfy all of the assumptions of Lemma* 2.1. *Suppose f satisfies* (3.1) *and is regularly varying: for some β, $\lim_{t\to\infty} f(tx)/f(t) = x^\beta$ for all $x > 0$.* (*Note that* (3.1) *entails $\beta \geqslant 0$.*) *If also*

$$f(t)^2/a(f(t)) = \mathrm{o}(t), \quad t\to\infty, \tag{3.13}$$

then (3.12) *holds, or, equivalently, see* (2.11),

$$\log \mathbf{P}[T_f > t] \sim -\mu_1 \int_0^t a(f(s))/f(s)^2\,\mathrm{d}s, \quad t\to\infty, \tag{3.14}$$

where μ_1 is the smallest eigenvalue of (2.13).

Remark 3.2. One may easily show, using elementary properties of regular variation, that (3.13) forces $\beta \leqslant 1/(2-\alpha)$ and that $\beta < 1/(2-\alpha)$ forces (3.13). Thus (3.13) is a genuine restriction only in the case $\beta = 1/(2-\alpha)$, however, it is indeed a restriction in this case. For example, Theorem 3.1 can not be applied in the case that $f(t) \sim t^{1/2}$ or $f(t) = (1-\varepsilon)(2t \log\log t)^{1/2}$ and X is standard Brownian motion. It does apply, though, if $f(t) = t^{1/2}S(t)$ with $S(t)\to 0$ as $t\to\infty$, S slowly varying.

Remark 3.3. Put $r = 1/(2-\alpha)$ and let γ_1 be the smallest positive root of $J_{-r}(\gamma_1) = 0$ where J_{-r} is the usual Bessel function of order $-r$. Then μ_1, the smallest eigenvalue of (2.13), is given by $\mu_1 = \gamma_1^2/4r^2$. The corresponding eigenfunction is $\psi_1(x) = c|x|^{1/2}J_{-r}(2r\mu_1^{1/2}|x|^{1/2r})$ where c is a normalizing constant (see [2, p. 156]). When $\alpha = 0$ we get $\mu_1 = \frac{1}{4}\pi^2$ and thus, in the case of standard Brownian motion ($a(x) \equiv \frac{1}{2}$), (3.14) takes on the form

$$\mathbf{P}[T_f > t] = \exp\left\{-\tfrac{1}{8}\pi^2 \int_0^t f(s)^{-2}\,\mathrm{d}s(1+\mathrm{o}(1))\right\}.$$

(See the papers in the supplementary reference list for stronger results in the Brownian motion case and the related case of random walks.)

Proof of Theorem 3.1. To prove Theorem 3.1 we only need to establish (3.5), (3.10) and (3.11). Let us write

$$f(t)=t^{\beta}S(t), \quad t>0,$$

where S is slowly varying. By definition of $f(t_n)=l_{n+1}$ we have, for all n,

$$f(t_n)/f(t_{n-1})=(t_n/t_{n-1})^{\beta}S(t_n)/S(t_{n-1})=1+\varepsilon.$$

From this and elementary properties of slowly varying functions (use [1, Lemma 2, p. 277]) it follows that as $n\to\infty$

$$t_n/t_{n-1}\to(1+\varepsilon)^{1/\beta}, \qquad \Delta_n/t_{n-1}\to(1+\varepsilon)^{1/\beta}-1. \tag{3.15}$$

From (3.13) and (3.15) we immediately get (3.5). Next, since $t\mapsto\lambda_1(f(t))$ is decreasing,

$$\int_{t_n}^{t_{n+1}}\lambda_1(f(s))\,\mathrm{d}s\Big/\int_0^{t_n}\lambda_1(f(s))\,\mathrm{d}s\leqslant\Delta_{n+1}/t_n \;\to\; (1+\varepsilon)^{1/\beta}-1$$

as $n\to\infty$, and letting $\varepsilon\to 0$ we get (3.10). It remains to establish (3.11). From (2.11) we see that (3.13) is equivalent to

$$t\lambda_1(f(t))\to\infty, \quad t\to\infty. \tag{3.16}$$

Noting from (3.15) that $t_n\geqslant\sigma^n$, $\sigma=(1+\varepsilon)^{1/2\beta}>1$, for all n sufficiently large and applying L'Hopital's rule, we get, as $n\to\infty$,

$$n\Big/\int_0^{t_n}\lambda_1(f(s))\,\mathrm{d}s\leqslant n\Big/\int_0^{\sigma^n}\lambda_1(f(s))\,\mathrm{d}s\sim(\lambda_1(f(\sigma^n))\sigma^n\log\sigma)^{-1}\to 0.$$

This gives (3.11) since c_2 does not depend on n. Thus, the proof of Theorem 3.1 is done.

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