Strong Approximations to Brownian Local Time †

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1. Introduction

A strong approximation (or strong invariance principle) of random walks to Brownian motion is the construction of random walks and Brownian motions on a single probability space so that the normalized random walks converge to the Brownian motions a.s. at some rate. Given such a strong approximation, it is natural to ask if the "local times" of the random walks also converge a.s. to the local times of the Brownian motions. The first such strong invariance principle for local times that also gave a rate was given in [R] for the case of simple random walk. Since then there have been many papers on this subject with the aim of weakening the assumptions on the walk and improving the rate. See [Bo2] and [CH] and the references therein. For recent results along these lines, see [BK2].

Our aim in this paper is slightly different. We are not concerned with obtaining optimal rates here, but rather giving the most general conditions under which a strong approximation for the local times holds. We assume that our random walks are mean 0 and variance 1, but no other moment conditions are

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assumed. Our main results (Theorems 3.2 and 4.5) say that one has a strong approximation for the local times whenever one has a strong invariance principle for the random walks, provided only that some very mild conditions hold. This is quite surprising because local time is a highly discontinuous functional on $C[0,1]$. In particular this shows that the local times of the random walks converge a.s. to those of the Brownian motions in the case of the Hungarian embedding. Our results also give a strong invariance principle for random walks having only $2+\varepsilon$ moments.

The conditions we impose are (a) that either the random walk be lattice or strongly nonlattice and (b) that the rate of convergence of the random walks to the Brownian motions not be too slow.

There are a number of interesting by–products of our results. We obtain weak uniform invariance principles for the local times of finite variance random walks in the lattice and strongly nonlattice case. The lattice case was already known ([Bo1]), while the strongly nonlattice case improves upon some results of Borodin (see [Bo2]). Our methods also give a weak uniform invariance principle for intersection local times with only second moments (this has also recently been proved by Rosen) and a strong uniform invariance principle for intersection local times, which is new. Finally, we show that the general theorems in [BK1] for weak uniform invariance principles for local times on curves hold under the assumption of finite variance when the dimension is 3 or less.

We use the letter c with or without subscripts to denote constants whose value is unimportant and may change from line to line.

2. A local central limit theorem

We begin with some estimates on the transition densities for random walks in $\mathbb{R}^d$. Let X_i be a sequence of i.i.d. random variables with mean 0 and nondegenerate covariance matrix Q. We assume the X_i are lattice–valued, and without loss of generality we assume the lattice to be $\mathbb{Z}^d$. Let $S_n = \sum_{i=1}^n X_i$ and let $P_n(x,y) = \mathbb{P}^x(S_n = y)$. For now let us assume the S_n are strongly aperiodic (see [Sp, D5.1]).

2.1. Proposition. *Let* $\gamma \in (0,1]$. *Then*

$$|P_n(0,x) - P_n(0,y)| \leq cn^{-d/2}(|x-y|/\sqrt{n})^\gamma. \tag{2.1}$$

Proof. Following Spitzer ([Sp, T7.9]),

$$\begin{aligned} &P_n(0,x) - P_n(0,y) \\ &\quad = cn^{-d/2} \int_{\alpha \in \sqrt{n}C} \varphi^n(\alpha/\sqrt{n})(e^{-ix\cdot\alpha/\sqrt{n}} - e^{-iy\cdot\alpha/\sqrt{n}})\, d\alpha, \end{aligned}$$

where φ is the characteristic function of X_i and C is the cube of side length 2π centered at the origin.

Let $\gamma \in (0,1]$ and note that $|e^{ia} - e^{ib}| \leq c|a-b|^\gamma$. Let

$\Delta = (|x-y|/\sqrt{n})^\gamma$. For $|\alpha| \le 1$, we have $|\varphi(\alpha/\sqrt{n})| \le 1$, and so

$$\int_{|\alpha|\le 1} \le c\Delta \int_{|\alpha|\le 1} |\alpha|^\gamma \, d\alpha \le c\Delta.$$

For r small and $|\alpha| \in [1, r\sqrt{n}]$, we have $|\varphi^n(\alpha/\sqrt{n})| \le \exp(-\alpha^t Q\alpha/4)$ (see [Sp, p. 77]), where α^t denotes the transpose of α. So

$$\int_{1\le|\alpha|\le r\sqrt{n}} \le c\Delta \int_{1\le|\alpha|\le r\sqrt{n}} |\alpha|^\gamma \exp(-\alpha^t Q\alpha/4)\, d\alpha \le c\Delta.$$

Since the X_i are strongly aperiodic, $|\varphi^n(\alpha/\sqrt{n})| \le (1-\delta)^n$ for some $\delta > 0$ if $|\alpha| \ge r\sqrt{n}$ (cf. [Sp, p. 77]). Hence for any $K > 0$ there exists c such that

$$\int_{r\sqrt{n}\le|\alpha|} \le c\Delta(1-\delta)^n \int_{r\sqrt{n}\le|\alpha|\le\pi\sqrt{n}} |\alpha|^\gamma \, d\alpha \le c\Delta n^{-K}.$$

Summing the three terms gives (2.1). ∎

A similar proof, following [Sp, P7.10], shows

$$\left| \frac{|x|^2}{n} P_n(0,x) - \frac{|y|^2}{n} P_n(0,y) \right| \le c n^{-d/2} (|x-y|/\sqrt{n})^\gamma. \tag{2.2}$$

Let $G(x,y) = \sum_{n=0}^\infty P_n(x,y)$.

2.2. Proposition. *Suppose $d = 3$. Let $\gamma \in (0,1)$. Then*

$$|G(0,x) - G(0,y)| \le c\frac{|x-y|}{(|x|\wedge|y|)^2} + c\frac{|x-y|^\gamma}{(|x|\wedge|y|)^{1+\gamma}}.$$

Proof. Suppose $|x| \le |y|$. Note

$$P_n(0,x) - P_n(0,y) = \frac{n}{|x|^2}\Big(\frac{|x|^2}{n}P_n(0,x) - \frac{|y|^2}{n}P_n(0,y)\Big) + \Big(\frac{|y|^2}{|x|^2} - 1\Big)P_n(0,y).$$

Using (2.2) and the fact that $P_n(0,y) \le n^{-1/2}|y|^2$ ([Sp, P7.10]),

$$\begin{aligned}\sum_{n=1}^{[|x|^2]} |P_n(0,x) - P_n(0,y)| &\le c\sum_{n=1}^{[|x|^2]} \frac{n}{|x|^2}\frac{1}{n^{3/2}}\frac{|x-y|^\gamma}{n^{\gamma/2}} \\ &\quad + c\frac{|y|^2 - |x|^2}{|x|^2}\sum_{n=1}^{[|x|^2]}\frac{1}{n^{1/2}|y|^2} \\ &\le c\frac{|x-y|^\gamma}{|x|^2}|x|^{1-\gamma} \\ &\quad + c\frac{(|y|-|x|)(|y|+|x|)}{|x|^2|y|^2}|x| \\ &\le c\frac{|x-y|^\gamma}{|x|^{1+\gamma}} + c\frac{|x-y|}{|x|^2}.\end{aligned}$$

Using Proposition 2.1,

$$\sum_{n=[|x|^2]}^{\infty} |P_n(0,x) - P_n(0,y)| \le c\frac{|x-y|^\gamma}{|x|^{1+\gamma}}.$$

Adding these two inequalities gives our result. ∎

Using the method of [Sp, p. 310], the assumption of strong aperiodicity may be dispensed with. Note Proposition 2.2 and [Sp, P26.1] together give us the conclusion of Corollary 3.3 of [BK1] when $d = 3$ under the assumption of finite variance only. By the proofs of [BK1], then, we get

2.3. Theorem. *Suppose $d \leq 3$, the X_i's have finite variance, and Hypothesis 6.1 of* [BK1] *holds for some $\beta \in (0,1)$. Then the conclusion of Proposition 6.3 of* [BK1] *holds.*

Remarks. (1) The importance of Theorem 2.3 to us here is that Theorem 7.4 of [BK1] holds with only second moments, i.e., the local times of the random walk converge weakly to the local times of Brownian motion, uniformly over all levels x. This had previously been proved in [Bo1].

(2) Theorem 2.3 also tells us that Theorem 7.8 of [BK1] holds with only second moments, i.e., the intersection local times of two independent random walks on $\mathbb{Z}^2$ or $\mathbb{Z}^3$ converges weakly to the intersection local times of two independent Brownian motions, uniformly over all levels x. While we were writing up the present results, we learned that Jay Rosen had independently proved a more general result about weak uniform invariance principles for intersection local times. This did not surprise us – it was through discussions with him that we came to believe Theorem 7.8 of [BK1] should hold with only second moments.

3. Local times – lattice case

We assume in this section that we have $\mathbb{Z}$–valued random walks with mean 0, variance 1. We also assume that our walks are strongly aperiodic – this assumption can be dispensed with by the use of [Sp, P5.1].

Define $\eta(x,n) = \sum_{j=1}^n 1_{\{x\}}(S_j)$, the number of times before n that the random walk hits x. Let $\mathcal{F}_j$ be the σ–field generated by $S_1, \ldots, S_j$. Let

$$L^n(x,t) = \frac{1}{\sqrt{n}} \eta(\sqrt{n}x, nt)$$

for $x \in \mathbb{Z}/\sqrt{n}$, $t \in \mathbb{Z}/n$, and let $L^n(x,t)$ be defined by linear interpolation for all other x and t.

3.1. Proposition. *Let $\delta \in (0, 1/2)$. Then there exist c_1, c_2 such that*

$$\mathbb{P}\Big(\sup_{x,y\in\mathbb{R}, t\le 1} |L^n(x,t) - L^n(y,t)|/(|x-y|^{1/2-\delta} \wedge 1) \ge \lambda\Big) \le c_1 \exp(-c_2\lambda) + n^{-7}$$

and

$$\sup_{x,y\in\mathbb{R}, t\le 1} |L^n(x,t) - L^n(y,t)|/(|x-y|^{1/2-\delta} \wedge 1) \le c\log n, \qquad a.s.$$

Proof. Fix $x, y \in \mathbb{Z}$, $x \ne y$, and define $A_k = (\eta(x,k) - \eta(y,k))/\sqrt{n}$.

By Proposition 2.1 and the translation invariance of the P_j,

$$|\mathbb{E}^z A_k| \le \frac{1}{\sqrt{n}} \sum_{j=1}^{k} |P_j(z,x) - P_j(z,y)| \le \frac{1}{\sqrt{n}} \sum_{j=1}^{k} \frac{c}{\sqrt{j}} \left(\frac{|x-y|}{\sqrt{j}}\right)^{\gamma} = c\frac{|x-y|^{\gamma}}{n^{\gamma/2}}. \tag{3.1}$$

Similarly, using [Sp, P7.9],

$$\mathbb{E}^z \eta(x,k) \le c\sqrt{n}.$$

Let $\Delta A_j = A_{j+1} - A_j$. Since $|x-y| \ge 1$,

$$\mathbb{E}^z \sum_{j=0}^{n-1} (\Delta A_j)^2 \le \frac{1}{n} \mathbb{E}^z \sum_{j=0}^{n-1} 1_{\{x,y\}}(S_j) = \frac{1}{n}(\mathbb{E}^z \eta(x,n) + \mathbb{E}^z \eta(y,n)) \le \frac{c}{\sqrt{n}} \le c\frac{|x-y|}{\sqrt{n}}. \tag{3.2}$$

Now

$$\mathbb{E}^z \sum_{j=0}^{n-1} (A_n - A_{j+1}) \Delta A_j = \mathbb{E}^z \left[\sum_{j=0}^{n-1} \mathbb{E}^z (A_n - A_{j+1} | \mathcal{F}_{j+1}) \Delta A_j \right] = \mathbb{E}^z \sum_{j=0}^{n-1} \mathbb{E}^{S_{j+1}} (A_{n-j-1}) \Delta A_j \tag{3.3}$$

$$\leq c\frac{|x-y|^\gamma}{n^{\gamma/2}}\mathbb{E}^z\sum_{j=0}^{n-1}|\Delta A_j|$$
$$\leq c\frac{|x-y|^\gamma}{n^{\gamma/2}}\frac{1}{\sqrt{n}}\mathbb{E}^z[\eta(x,n)+\eta(y,n)]$$
$$\leq c\bigg(\frac{|x-y|}{\sqrt{n}}\bigg)^\gamma.$$

Since

$$A_n^2 = 2\sum_{j=0}^{n-1}(A_n - A_{j+1})\Delta A_j + \sum_{j=0}^{n-1}(\Delta A_j)^2,$$

(cf. [BK1, Eq. (5.3)]), combining (3.2) and (3.3) gives

$$\sup_z \mathbb{E}^z A_n^2 \leq c\Delta,$$

where

$$\Delta = \bigg(\frac{|x-y|}{\sqrt{n}}\bigg)^\gamma + \bigg(\frac{|x-y|}{\sqrt{n}}\bigg).$$

Let

$$U_j = \mathbb{E}^z[A_n - A_j|\mathcal{F}_j] = \mathbb{E}^{S_j}A_{n-j}.$$

By (3.1), $|U_j| \leq c\Delta$. Since $M_j = U_j + A_j$ is a martingale, Doob's inequality tells us that

$$\begin{aligned}\mathbb{E}^z(\sup_{j\le n}|A_j|)^2 &\le 2\mathbb{E}^z(\sup_{j\le n}|U_j|)^2+2\mathbb{E}^z(\sup_{j\le n}|M_j|)^2\\ &\le c\Delta^2+8\mathbb{E}^z M_n^2\\ &\le c\Delta^2+16\mathbb{E}^z U_n^2+16\mathbb{E}^z A_n^2\\ &\le c(\Delta^2+\Delta).\end{aligned}$$

Hence for any $\mathcal{F}_j$–stopping time N bounded by n,

$$\mathbb{E}^z[(A_n-A_N)^2|\mathcal{F}_N]\le \mathbb{E}^{S_N}[\sup_{j\le n}|A_j|^2]\le c(\Delta^2+\Delta).$$

Using Cauchy–Schwarz, the fact that $\sup_{j\le n-1}|\Delta A_j|\le 1/\sqrt{n}$, and [DM, p. 193],

$$\mathbb{P}^z(\sup_{k\le n}|A_k|\ge\lambda)\le c_1\exp(-c_2\lambda/(\Delta+\Delta^2)^{1/2}) \tag{3.4}$$

(cf. [BK1, Proposition 5.2]).

A standard metric entropy argument using (3.4) shows that for each $\delta>0$,

$$\begin{aligned}\mathbb{P}^z(\sup_{|x|,|y|\le n^4,t\le 1}|L^n(x,t)-L^n(y,t)|/(|x-y|^{1/2-\delta}\wedge 1)>\lambda)&\\ \le c_1\exp(-c_2\lambda).&\end{aligned}$$

Using the fact that

$$\mathbb{P}^0(\sup_{j\le n}|S_j|\ge n^4)\le n^{-7} \tag{3.5}$$

by Chebyshev's inequality, we get the first assertion. A standard Borel–Cantelli argument (with $\lambda_n = c \log n$) shows that there exists c such that

(3.6)
$$\sup_{x,y\in\mathbb{R},t\le 1} |L^n(x,t) - L^n(y,t)|/(|x-y|^{1/2-\delta} \wedge 1) \le c \log n, \quad \text{a.s.}$$

This completes the proof. ∎

We are now ready for the main result of this section. Suppose for each n we have i.i.d. random variables $X_1^n, \dots, X_n^n$, and we set $S_k^n = \sum_{j=1}^k X_j^n$. Define η^n in terms of the S_j^n analogously to the definition of η above and let us now use $L^n(x,t)$ to denote $\eta^n(\sqrt{n}x, nt)/\sqrt{n}$ for $x \in \mathbb{Z}/\sqrt{n}$, $t \in \mathbb{Z}/n$ with linear interpolation for all other x and t. For each n let B_t^n be a Brownian motion with local times $\ell^n(x,t)$.

3.2. Theorem. *Let $\sigma > 0$. Suppose a_n is a sequence such that for some $\delta > 0$, $a_n \le (\log n)^{-6}$. Suppose*

$$\sup_{t\le 1} |S_{[nt]}^n/\sqrt{n} - B_t^n| = O(a_n), \quad \textit{a.s.}$$

Then

$$\sup_{x\in\mathbb{R},t\le 1} |L^n(x,t) - \ell^n(x,t)| = O(n^{-1/(5+\sigma)} \wedge a_n^{1/(5+\sigma)} \log n), \quad \textit{a.s.}$$

Remarks. (1) The statement of our theorem includes the case where the X_j^n do not depend on n, B_t is a single Brownian motion, and $B^n_t = B_{nt}/\sqrt{n}$, the situation that comes up in Skorokhod embedding.

(2) Much better rates are possible when the X_i satisfy higher moment conditions. See [BK2].

(3) If the X_j have $2+\varepsilon$ moments, then any reasonable Skorokhod embedding will have a_n satisfying the above hypotheses.

(4) The Hungarian embedding satisfies the above with $a_n = \log n/\sqrt{n}$.

Proof. First note that (3.6) holds for the new $L^n(x,t)$ as well as the old ones, since the estimates (3.4) and (3.5) are valid with constants independent of n. Similarly we get the estimates

$$\sup_{t\le 1} |\ell^n(x,t) - \ell^n(y,t)| \le c(|x-y|^{1/2-\delta} \wedge 1)\log n, \quad \text{a.s.} \tag{3.7}$$

and

$$\sup_{x\in\mathbb{R}, t\le 1} L^n(x,t) \le c\log n, \quad \text{a.s.} \tag{3.8}$$

We now fix an ω not in any of the exceptional sets and proceed to get a bound on $|L^n(x,t) - \ell^n(x,t)|$ independent of x and t. For simplicity of notation we will do the case $t = 1$. Fix x_0. Define $f_\varepsilon(x_0) = 1/\varepsilon$, define $f_\varepsilon(y) = 0$ if $|y - x_0| \ge \varepsilon$, and define it by linear interpolation for $y \in (x_0 - \varepsilon, x_0)$ and $(x_0, x_0 + \varepsilon)$. Take δ small.

Since $\int f_\varepsilon(y)\,dy = 1$,

$$
\begin{aligned}
|\int_0^1 f_\varepsilon(B^n_s)\,ds - \ell^n(x_0,1)| &= |\int f_\varepsilon(y)\ell^n(y,1)\,dy - \ell^n(x_0,1)| \\
&\le \sup_{|y-x_0|\le\varepsilon} |\ell^n(y,1) - \ell^n(x_0,1)| \qquad (3.9)\\
&\le c\varepsilon^{1/2-\delta}\log n.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&|\int_0^1 f_\varepsilon(B^n_s)\,ds - \frac{1}{n}\sum_{j=1}^n f_\varepsilon(S_j/\sqrt{n})| \\
&\qquad \le |\int_0^1 f_\varepsilon(B^n_s)\,ds - \int_0^1 f_\varepsilon(S_{[ns]}/\sqrt{n})\,ds| + 2/\varepsilon n \qquad (3.10)\\
&\qquad \le \|f'_\varepsilon\|_\infty a_n + 2/\varepsilon n = a_n/\varepsilon^2 + 2/\varepsilon n.
\end{aligned}
$$

Now

$$
\begin{aligned}
\frac{1}{n}\sum_{j=1}^n f_\varepsilon(S_j/\sqrt{n}) &= \frac{1}{n}\sum_{y\in\mathbb{Z}/\sqrt{n}} f_\varepsilon(y)\eta^n(\sqrt{n}y,n) \\
&= \frac{1}{\sqrt{n}}\sum_{y\in\mathbb{Z}/\sqrt{n}} f_\varepsilon(y)[L^n(y,1) - L^n(x_0,1)] \\
&\qquad + L^n(x_0,1)\left[\frac{1}{\sqrt{n}}\sum_{y\in\mathbb{Z}/\sqrt{n}} f_\varepsilon(y) - 1\right] + L^n(x_0,1).
\end{aligned}
$$

It is easy to see that $|n^{-1/2}\sum_{y\in\mathbb{Z}/\sqrt{n}} f_\varepsilon(y) - 1| \le 1/n\varepsilon$. So using (3.6) and (3.8),

$$
|\frac{1}{n}\sum_{j=1}^n f_\varepsilon(S_j/\sqrt{n}) - L^n(x_0,1)|
$$

$$\leq c\varepsilon^{1/2-\delta}\log n + c\log n/\varepsilon n. \tag{3.11}$$

Adding (3.9), (3.10) and (3.11) we get that

$$|L^n(x_0,1) - \ell^n(x_0,1)| \leq c\varepsilon^{1/2-\delta}\log n + a_n/\varepsilon^2 + c\log n/\varepsilon n.$$

Now choose $\varepsilon = \varepsilon_n = a_n^{2/5} \vee n^{-1/2}$. ∎

3.3. Corollary. *Let $\sigma > 0$. If $a_n \to 0$ and*

$$\sup_{t\leq 1} |S^n_{[nt]}/\sqrt{n} - B^n_t| = O(a_n) \quad \textit{in probability,}$$

then

$$\sup_{x\in\mathbb{R}, t\leq 1} |L^n(x,t) - \ell^n(x,t)| = O(a_N^{1/(5+\sigma)} \wedge n^{-1/(5+\sigma)}) \quad \textit{in probability.}$$

Proof. Let δ be chosen small. Let $\zeta > 0$. Provided the constants c are chosen large enough, we have by the remarks preceding Theorem 3.2 and the proof of Theorem 3.2 that for all n sufficiently large,

$$\mathbb{P}(\sup_{x,y\in\mathbb{R}, t\leq 1} |L^n(x,t) - L^n(y,t)|/(|x-y|^{1/2-\delta} \wedge 1) > c) < \zeta/3,$$

$$\mathbb{P}(\sup_{x,y\in\mathbb{R}, t\leq 1} |\ell^n(x,t) - \ell^n(y,t)|/(|x-y|^{1/2-\delta} \wedge 1) > c) < \zeta/3$$

and

$$\mathbb{P}(\sup_{t\le 1, x\in\mathbb{R}} |L^n(x,t)| > c) < \zeta/3.$$

So, following the proof of Theorem 3.2, for each n sufficiently large there exists a set N_n of probability less than ζ such that if $\omega \notin N_n$, then

$$|L^n(x_0,1) - \ell^n(x_0,1)| \le c\varepsilon^{1/2-\delta} + a_n/\varepsilon^2 + c/\varepsilon n$$

(the $\log n$'s are no longer present). Now choose ε_n as before. ∎

Remark. If one has a lattice valued random walk in two or three dimensions, strong approximations to 2 or 3 dimensional Brownian motion exist (see [E]). A proof very similar to the one above shows that if two independent random walks converge uniformly to two independent Brownian motions a.s. at a sufficiently fast rate a_n, then the intersection local times of the two random walks will converge a.s. to the intersection local times of the two Brownian motions, uniformly over all levels x, at a rate of a_n to a suitable power.

4. Local times – nonlattice case

In this section we assume the X_i are i.i.d. $\mathbb{R}$–valued random variables with mean 0, variance 1 and are nonlattice, i.e., $|\varphi(\theta)| < 1$ for all $\theta \ne 0$. We also make the assumption throughout this section that the X_i are strongly nonlattice, that is,

$$\limsup_{|\theta|\to\infty} |\varphi(\theta)| < 1.$$

Since $|\varphi(\theta)| < 1$ for all $\theta \neq 0$, we get in this case that for any $r > 0$

$$\rho_r = \sup_{|\theta|\geq r} |\varphi(\theta)| < 1.$$

The assumption that the X_i are strongly nonlattice is reasonably weak, being satisfied, for example, if the distribution of X_i has a nonzero absolutely continuous part. (This follows using the Riemann–Lebesgue lemma.) In particular, Borodin's condition [Bo2], that $\varphi \in L^2$, implies that S_2 has a bounded density, hence that $\varphi(\theta)^2 \to 0$ as $|\theta| \to \infty$. See also Section 4 of [BK2].

We proceed to get two estimates of local limit type. Stone [St] has similar results without the strongly nonlattice assumption, but they are not sufficiently strong enough for our purposes here.

Let I_h denote $[-h, h]$.

4.1. Proposition. *Let $K \in \mathbb{Z}^+$. Then for all x, z*

$$\mathbb{P}^z(S_n \in x + I_h) \leq c\frac{h}{\sqrt{n}} + c\frac{1}{n^K}.$$

Proof. Let $\Gamma_h(y) = e^{1/2}\exp(-y^2/2h^2)$. Then

$$\begin{aligned}\mathbb{P}^z(S_n \in x + I_h) &\le \mathbb{E}^z\Gamma_h(S_n - x)\\ &= c\int_{\mathbb{R}} e^{-iz\theta}e^{ix\theta}\widehat{\Gamma}_h(\theta)[\varphi(\theta)]^n\,d\theta\\ &\le \frac{c}{\sqrt{n}}\int_{\mathbb{R}}\widehat{\Gamma}_h(\alpha/\sqrt{n})|\varphi(\alpha/\sqrt{n})|^n\,d\alpha,\end{aligned}$$

where $\widehat{\Gamma}_h$ denotes the Fourier transform of Γ_h; thus $\widehat{\Gamma}_h = ch\exp(-\theta^2h^2/2) > 0$.

First,

$$\int_{|\alpha|\le 1} \le \frac{\|\widehat{\Gamma}_h\|_\infty}{\sqrt{n}}\int_{|\alpha|\le 1} d\alpha \le \frac{ch}{\sqrt{n}}.$$

As in the proof of Proposition 2.1, if r is small, $|\varphi(\alpha/\sqrt{n})|^n \le \exp(-\alpha^2/4)$ for $1 \le |\alpha| \le r\sqrt{n}$, so

$$\int_{1\le|\alpha|\le r\sqrt{n}} \le \frac{\|\widehat{\Gamma}_h\|_\infty}{\sqrt{n}}\int_{|\alpha|\le r\sqrt{n}}\exp(-\alpha^2/4)\,d\alpha \le \frac{ch}{\sqrt{n}}.$$

Finally,

$$\begin{aligned}\int_{|\alpha|\ge r\sqrt{n}} &= \sqrt{n}\int_{|\theta|\ge r}\widehat{\Gamma}_h(\theta)|\varphi(\theta)|^n\,d\theta\\ &\le \sqrt{n}\rho_r^n\int\widehat{\Gamma}_h(\theta)\,d\theta \le c\sqrt{n}\rho_r^n.\end{aligned}$$

Adding these three inequalities gives our result. ∎

Now let $\beta \in (0,1)$ and let ψ_β be a nonnegative C^∞ function with

$$1_{[-\frac{1}{2}-\beta,\frac{1}{2}+\beta]} \geq \psi_\beta \geq 1_{[-\frac{1}{2},\frac{1}{2}]}, \quad \|\psi'_\beta\|_\infty \leq c/\beta,$$
$$\|\psi''_\beta\|_\infty \leq c/\beta^2.$$

4.2. Proposition. *Let $K \in \mathbb{Z}^+$. Then for all x, y, z*

$$|\mathbb{E}^z \psi_\beta(S_n - x) - \mathbb{E}^z \psi_\beta(S_n - y)| \leq c\left(\frac{|x-y|}{\sqrt{n}}\right)^\gamma \left(\frac{1}{\sqrt{n}} + \frac{1}{\beta n^K}\right).$$

Proof. Letting $\widehat{\psi}_\beta$ denote the Fourier transform of ψ_β, note that $\|\widehat{\psi}_\beta\|_\infty \leq \|\psi_\beta\|_1 \leq 3$ and

$$|\theta^2 \widehat{\psi}_\beta(\theta)| = |\widehat{\psi}''_\beta(\theta)| \leq \|\psi''_\beta\|_1 \leq c/\beta.$$

It follows that $|\widehat{\psi}_\beta(\theta)| \leq c/\beta\theta^2$, and so $\int_{|\theta|\geq r} |\theta|^\gamma |\widehat{\psi}_\beta(\theta)|\, d\theta \leq c/\beta$ if $\gamma \in (0,1)$, $r > 0$.

With these estimates on $\|\widehat{\psi}_\beta\|_\infty$ and $\|\widehat{\psi}_\beta\|_1$, the proof is exactly the same as the proof of Proposition 4.1, except that just as in the proof of Proposition 2.1, we use

$$|e^{ix\alpha/\sqrt{n}} - e^{iy\alpha/\sqrt{n}}| \leq |\alpha|^\gamma (|x-y|/\sqrt{n})^\gamma$$

to get the extra $(|x-y|/\sqrt{n})^\gamma$ term. ∎

Define

$$\eta(x,k) = \sum_{j=1}^k 1_{[-1/2,1/2)}(S_j - x), \qquad \widetilde{\eta}_\beta(x,k) = \sum_{j=1}^k \psi_\beta(S_j - x),$$

and

$$L^n(x,t) = \frac{1}{\sqrt{n}}\eta(x\sqrt{n}, nt), \qquad \widetilde{L}^n_\beta(x,t) = \frac{1}{\sqrt{n}}\widetilde{\eta}_\beta(x\sqrt{n}, nt).$$

4.3. Proposition. *Let $\delta \in (0, 1/2)$. Suppose $\beta n^{K-1} \geq 1$. Then there exist c_1, c_2 such that*

$$\mathbb{P}(\sup_{x,y\in\mathbb{R}, t\leq 1} |\widetilde{L}^n_\beta(x,t) - \widetilde{L}^n_\beta(y,t)|/(|x-y|^{1/2-\delta} \wedge 1) \geq \lambda/\beta)$$
$$\leq c_1 \exp(-c_2\lambda) + n^{-7}$$

and

$$\sup_{x,y\in\mathbb{R}, t\leq 1} |\widetilde{L}^n_\beta(x,t) - \widetilde{L}^n_\beta(y,t)|/(|x-y|^{1/2-\delta} \wedge 1) \leq \frac{c}{\beta}\log n, \quad a.s.$$

Proof. Let

$$A_k = \frac{1}{\sqrt{n}}(\widetilde{\eta}_\beta(x,k) - \widetilde{\eta}_\beta(y,k)).$$

By Proposition 4.2,

$$|\mathbb{E}^z A_k| \leq c\left(\frac{|x-y|}{\sqrt{n}}\right)^\gamma.$$

Note that if $|x-y| \geq 1$,

$$\mathbb{E}^z \sum_{j=0}^{n-1} (\Delta A_j)^2 \leq \frac{1}{n} \mathbb{E}^z \sum_{j=0}^{n-1} \Big(1_{[x-1,x+1]}(S_j) + 1_{[y-1,y+1]}(S_j) \Big)$$
$$\leq \frac{c}{\sqrt{n}} \leq \frac{c|x-y|}{\sqrt{n}\beta}$$

by using Proposition 4.1.

On the other hand, if $|x-y| \leq 1$,

$$|\psi_\beta(S_j - x) - \psi_\beta(S_j - y)| \leq \|\psi_\beta'\|_\infty |x-y| \leq c|x-y|/\beta$$

and

$$\mathbb{E}^z \sum_{j=0}^{n-1} (\Delta A_j)^2 = \frac{1}{n} \mathbb{E}^z \sum_{j=0}^{n-1} [\psi_\beta(S_j - x) - \psi_\beta(S_j - y)]^2$$
$$\leq \frac{c|x-y|}{\sqrt{n}\beta} \mathbb{E}^z \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} [\psi_\beta(S_j - x) + \psi_\beta(S_j - y)]$$
$$\leq \frac{c|x-y|}{\sqrt{n}\beta} \mathbb{E}^z \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} [1_{[x-1,x+1]}(S_j) + 1_{[y-1,y+1]}(S_j)]$$
$$\leq \frac{c|x-y|}{\sqrt{n}\beta}.$$

Similarly,

$$\sup_{j \leq n-1} |\Delta A_j|^2 \leq \frac{c|x-y|}{\beta n}.$$

With these estimates in place of (3.2), we may now proceed exactly as in Proposition 3.1 to obtain our result. ∎

To go from $\widetilde{L}^n_\beta(x,t)$ to $L^n(x,t)$ we have

4.4. Proposition. *If* $\beta_n \to 0$ *so that* $\liminf \log \beta_n / \log n > -\infty$*, then there exists* c *such that*

$$(a) \qquad \sup_{x\in\mathbb{R}, t\le 1} |\widetilde{L}^n_{\beta_n}(x,t) - L^n(x,t)| \le c\beta_n \log n, \quad a.s.$$

and

$$(b) \qquad \sup_{x\in\mathbb{R}} \frac{1}{\sqrt{n}} \sum_{j=1}^n 1_{[x,x+\beta_n]}(S_j) \le c\beta_n \log n, \quad a.s.$$

Proof. If I is an interval of length $2\beta_n$,

$$\sup_x \mathbb{P}^x(S_j \in I) \le c\beta_n/\sqrt{j} + c/j^K$$

by Proposition 4.1. Summing,

$$\sup_x \mathbb{E}^x \sum_{j=1}^n 1_I(S_j) \le c\beta_n\sqrt{n} + c/n^{K-1}.$$

By our assumption on β_n, this is less than $c\beta_n\sqrt{n}$. The quantity $A_k = \sum_{j=1}^k 1_I(S_j)$ is a subadditive functional. By Chebyshev's inequality,

$$\sup_x \mathbb{P}^x(A_n > 2c\beta_n\sqrt{n}) \le 1/2.$$

So by the strong Markov property,

$$\mathbb{P}^x(A_n > 2mc\beta_n\sqrt{n}) \le (1/2)^m.$$

Therefore

$$\mathbb{P}^0\bigg(\frac{1}{\sqrt{n}}\sum_{j=1}^n 1_I(S_j) \ge c_1\beta_n \log n/2\bigg) \le \exp(-c_1c_2\log n).$$

If $|\widetilde{L}^n_{\beta_n}(x,t) - L^n(x,t)| \ge c_1\beta_n\log n$, for some x, then either (i) $\sup_{j\le n}|S_j| \ge n^4$, or (ii) for some interval I of length β_n contained in $[-n^4-1, n^4+1]$, we have $(1/\sqrt{n})\sum_{j=1}^n 1_I(S_j) \ge c_1\beta_n\log n/2$. Any interval of length β_n is contained in some interval $[k\beta_n, (k+2)\beta_n]$ for some integer k. There are at most $2n^4/\beta_n$ such intervals contained in $[-n^4-1, n^4+1]$. So the probability of possibility (ii) is bounded by $(2n^4/\beta_n)\exp(-c_1c_2\log n)$. The probability of possibility (i) is bounded by n^{-7} by Chebyshev's inequality. The result (a) follows by Borel–Cantelli if we use the assumption on β_n and take c_1 large enough. The same proof also shows (b). ∎

Now define the triangular arrays X^n_j, the partial sums, the local times, and the Brownian motions similarly to what was done in Section 3.

4.5 Theorem. *Let $\sigma > 0$. Suppose a_n is a sequence such that $a_n \le (\log n)^{-10}$. Suppose*

$$\sup_{t\le 1}|S^n_{[nt]}/\sqrt{n} - B^n_t| = O(a_n), \quad \textit{a.s.}$$

Then

$$\sup_{x\in\mathbb{R},t\le 1}|L^n(x,t)-\ell^n(x,t)|$$
$$= O(n^{-1/(9+\sigma)} \vee a_n^{1/(9+\sigma)} \log n), \quad a.s.$$

Proof. The proof follows the lines of the proof of Theorem 3.2 closely, but with the following changes. Let $f_{\varepsilon n}$ be the step function which takes the value $f_\varepsilon(x_0+k/\sqrt{n})$ for $x\in[x_0+k/\sqrt{n}, x_0+(k+1)/\sqrt{n})$, $k\in\mathbb{Z}$. Then

$$\left|\frac{1}{n}\sum_{j=1}^{n} f_\varepsilon(S_j/\sqrt{n}) - \frac{1}{n}\sum_{j=1}^{n} f_{\varepsilon n}(S_j/\sqrt{n})\right|$$
$$\le \|f_\varepsilon - f_{\varepsilon n}\|_\infty \le 1/\varepsilon\sqrt{n}.$$

Note that

$$\frac{1}{n}\sum_{j=1}^{n} f_{\varepsilon n}(S_j/\sqrt{n}) = \frac{1}{n}\sum_{y\in\mathbb{Z}/\sqrt{n}} f_\varepsilon(y)\eta^n(\sqrt{n}y+\frac{1}{2},n).$$

Also, by Propositions 4.3 and 4.4, if $|y-x_0|\le\varepsilon$,

$$|L^n(y,1)-L^n(x_0,1)| \le \frac{c}{\beta_n}\varepsilon^{1/2-\delta}\log n + c\beta_n\log n.$$

With these changes to (3.11), we get

$$|L^n(x_0,1)-\ell^n(x_0,1)| \le \frac{c\varepsilon^{1/2-\delta}}{\beta_n}\log n + \frac{a_n}{\varepsilon^2}$$
$$+\frac{c\log n}{\varepsilon n} + \frac{1}{\varepsilon\sqrt{n}} + c\beta_n\log n.$$

Now choose $\varepsilon = \varepsilon_n = a_n^{4/9} \vee n^{-1/4}$ and $\beta_n = \varepsilon^{1/4-\delta/2}$. ∎

Returning to a single sequence S_n and a single Brownian motion B_t with local times $\ell(x,t)$, we also have a uniform weak invariance principle.

4.6. Theorem. *For mean 0, variance 1 strongly nonlattice random walks,*

$$\{(S_{[nt]}/\sqrt{n}, L^n(x,t)) : t \le 1, x \in \mathbb{R}\}$$

converges weakly to

$$\{(B_t, \ell(x,t)) : t \le 1, x \in \mathbb{R}\}.$$

Proof. One can find another probability space which contains random walks with the same distribution as S_n and a Brownian motion such that the normalized random walks converge a.s. to the Brownian motion, uniformly over $t \le 1$. In particular, they converge in probability.

We now imitate the proofs of Theorem 4.5 and Corollary 3.3 to conclude that the $L^n(x,t)$ converge in probability to $\ell^n(x,t)$, uniformly over $t \le 1, x \in \mathbb{R}$. This proves weak convergence. ∎

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