Stable-like processes

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Many real-world phenomena are better explained by models that have both a continuous part and jumps. For example, although the stock market is usually modeled by geometric Brownian motion, a Brownian motion cannot always move fast enough.

To study processes that have both a continuous part and a jump part, it is plausible to consider both parts separately and understand each well.

The most basic continuous process is Brownian motion. The infinitesimal generator of Brownian motion is

$$\mathcal{L}f(x) = \frac{1}{2}\Delta f(x).$$

The most basic jump process is the Poisson process.

A step up in complexity is the class of Lévy processes. The infinitesimal generator of a Lévy process is

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} [f(x+h) - f(x) - 1_{(|h| \le 1)} \nabla f(x) \cdot h] \, n(dh),$$

where

$$\int_{\mathbb{R}^d} (1 \wedge |h|^2) \, n(dh) < \infty.$$

To generalize Brownian motion to other continuous diffusions, two classes of operators are common.

Non-divergence form operators

Non-divergence form operators have the form

$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial f}{\partial x_i}(x).$$

We assume the a_{ij} are symmetric, uniformly positive definite, bounded, and $x \to a_{ij}(x)$ is measurable.

To generalize Lévy processes, one looks at

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} [f(x+h) - f(x) - 1_{(|h|\le 1)} \nabla f(x) \cdot h] \, n(x, dh),$$

with an appropriate set of conditions on n.

Divergence form operators

The other common type of operator associated to continuous diffusions is that of divergence form. These have the form

$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(\cdot)\frac{\partial f}{\partial x_j}(\cdot)\Big)(x).$$

To even make sense of this when a_{ij} is not differentiable but only bounded and measurable, multiply $\mathcal{L}f(x)$ be $g(x)$, integrate, and use integration by parts:

$$\int_{\mathbb{R}^d} g(x)(\mathcal{L}f(x))\,dx = -\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} a_{ij}(x)\frac{\partial f}{\partial x_i}(x)\frac{\partial g}{\partial x_j}(x)\,dx.$$

Dirichlet forms

We call

$$\mathcal{E}(f,g) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_j}(x)\, dx$$

the Dirichlet form associated with $\mathcal{L}$.

For jump processes, the Dirichlet forms one looks at have the form

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))(g(y) - g(x))\, J(dx, dy),$$

where J is a symmetric measure satisfying some integrability conditions.

The two classes of jump processes I talked about are too general to get good results, so let's specialize.

I will talk about 4 models.

(1) Infinitesimal generator, fixed order.

(2) Dirichlet form

(3) Infinitesimal generator, variable order.

(4) Stochastic differential equation

For operators of the form

$$\mathcal{L}f(x) = \int [f(x+h) - f(x) - 1_{(|h|\le 1)} \nabla f(x) \cdot h] \, n(x, dh),$$

let's suppose

$$n(x, dh) = n(x, h)\, dh$$

for some function n.

When

$$n(x,h) = \frac{c}{|h|^{d+\alpha}},$$

then the corresponding process is the symmetric stable process of order α. Let's look at n's of the form

$$n(x,h) = \frac{A(x,h)}{|h|^{d+\alpha}},$$

where $A(x,h)$ is bounded above and below.

This operator is of fixed order.

The corresponding process is an example of a stable-like process.

For processes associated with a Dirichlet form of the form

$$\mathcal{E}(f,g) = \int\int (f(y)-f(x))(g(y)-g(x))\,J(dx,dy),$$

when

$$J(dx,dy) = \frac{c}{|x-y|^{d+\alpha}}\,dx\,dy,$$

we again have a symmetric stable process of order α.

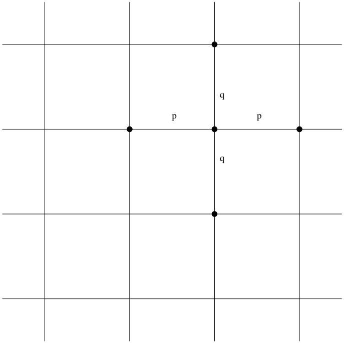
Another type of stable-like process is the one associated with the Dirichlet form

$$\mathcal{E}(f,g) = \int\int (f(y)-f(x))(g(y)-g(x))\,J(dx,dy),$$

where

$$J(dx,dy) = \frac{B(x,y)}{|x-y|^{d+\alpha}}$$

and B is symmetric and bounded above and below.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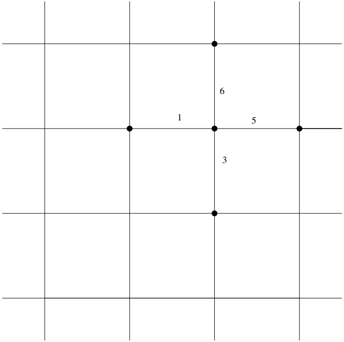
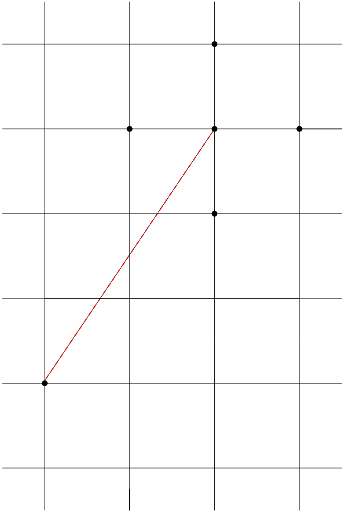


The probability of moving vertically is

$$\frac{6}{6+1+5+3} = \frac{6}{15} = \frac{2}{5}.$$



Let me mention two other types of stable-like processes that have been studied. One class corresponds to the operator

$$\mathcal{L}f(x) = \int [f(x+h) - f(x) - 1_{(|h|\le 1)} \nabla f(x) \cdot h] \, n(x, dh),$$

where

$$n(x, dh) = \frac{c}{|h|^{d+\alpha(x)}} \, dh.$$

This is an example of a variable order operator.

SDEs

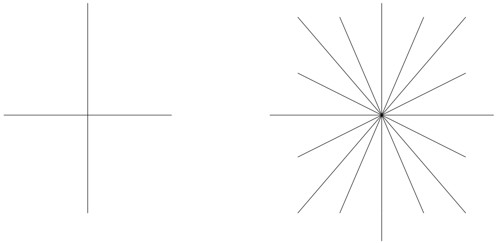
The other type can be described via stochastic differential equations. Let $Z^1, \ldots, Z^d$ be i.i.d. one-dimensional stable processes of order α. Consider the solution to

$$dX_t = A(X_{t-})\, dZ_t, \qquad X_0 = x_0,$$

which means

$$X_t^i = x_0^i + \int_0^t \sum_{j=1}^d A_{ij}(X_{s-})\, dZ_s^j.$$

The d-dimensional process $(Z^1, \ldots, Z^d)$ is not the same as a symmetric stable process of order α.



Uniqueness

Let me mention some uniqueness results. For the SDE

$$dX_t = A(X_{t-})\, dZ_t,$$

there is uniqueness in law if the matrix A is continuous in x and non-degenerate at every point.

For the operators $\mathcal{L}$, uniqueness is expressed in terms of the martingale problem: find a probability measure $\mathbb{P}$ such that $\mathbb{P}(X_0 = x_0) = 1$ and

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)\, ds$$

is a $\mathbb{P}$-local martingale if $f \in C^2(\mathbb{R}^d)$. Here X is the canonical process.

When

$$\mathcal{L}f(x) = \int [f(x+h) - f(x) - 1_{(|h|\le 1)} \nabla f(x) \cdot h]\, n(x, dh),$$

and

$$n(x, dh) = \frac{c}{|h|^{d+\alpha(x)}}\, dh,$$

there is uniqueness for the martingale problem when $\alpha(x)$ is Dini continuous.

When

$$\mathcal{L}f(x) = \int [f(x+h) - f(x) - 1_{(|h|\le 1)} \nabla f(x) \cdot h] \, n(x, dh),$$

and

$$n(x, dh) = \frac{A(x,h)}{|h|^{d+\alpha}} \, dh,$$

a sufficient condition for uniqueness for the martingale problem is that

$$\sup_{|h|\le \frac{1}{2}} |A(x,h) - A(y,h)| \log^2(1/|h|) \to 0$$

as $y \to x$.

It is very probable that some continuity is required.

Uniqueness is not an issue when discussing processes associated to Dirichlet forms.

Harnack inequalities

What about regularity results? One regularity result that one would like is the Harnack inequality: if h is harmonic in $B(x_0, 2R)$ and non-negative,

$$h(x) \le ch(y), \qquad x, y \in B(x_0, R).$$

One definition of harmonic is that $\mathcal{L}h = 0$ in $B(x_0, 2R)$. A more general definition is that $h(X_{t\wedge\tau})$ is a $\mathbb{P}^x$-martingale for each x, where τ is the exit time for the ball $B(x_0, 2R)$.

This holds for stable-like processes associated to an operator $\mathcal{L}$ with

$$n(x, dh) = \frac{A(x, h)}{|h|^{d+\alpha}}\, dh$$

and to processes associated to a Dirichlet form $\mathcal{E}$ with

$$J(dx, dy) = \frac{B(x, y)}{|x - y|^{d+\alpha}}\, dx\, dy.$$

It is crucial that h be non-negative on all of $\mathbb{R}^d$, and also some boundedness condition is necessary.

The proofs of these two cases are quite different.

The Harnack inequality can fail for the operators associated to the solution to the SDE, even when $A(x)$ is equal to the identity for every x.

Hölder continuity

Another type of regularity one would like is the Hölder continuity of harmonic functions. If h is harmonic and bounded in $B(x_0, 2R)$, one would like the estimate

$$|h(x) - h(y)| \le c\Big(\sup_{B(x_0,2R)} |h|\Big)\Big(\frac{|x-y|}{R}\Big)^{\beta}$$

when $x, y \in B(x_0, R)$.

The Hölder regularity holds for the stable-like processes associated with the operator $\mathcal{L}$,

$$n(x, dh) = \frac{A(x, h)}{|h|^{d+\alpha}} \, dh.$$

It holds for the processes associated with the Dirichlet form with

$$J(dx, dy) = \frac{B(x, y)}{|x - y|^{d+\alpha}} \, dx \, dy.$$

Hölder regularity holds for the stable-like process given as the solution of the SDE.

Transition densities

Can one say anything about the transition densities? For the processes associated with the Dirichlet form, we have

$$p(t, x, y) \approx t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}.$$

Further regularity

Consider the stable-like processes associated with the operator $\mathcal{L}$ where

$$n(x, dh) = \frac{A(x,h)}{|h|^{d+\alpha}}\, dh.$$

Theorem. Suppose $f \in C^\beta$,

$$\sup_h |A(x,h) - A(y,h)| \le c|x-y|^\beta,$$

and

$$\beta, \alpha + \beta \notin \mathbb{N}.$$

Let

$$S_\lambda f(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt.$$

Then

$$\|S_\lambda f\|_{C^{\alpha+\beta}} \le c\|f\|_{C^\beta}.$$

Future work

(1) Dirichlet problem for the ball, both interior and boundary estimates.

(2) Estimates on transition densities.

(3) Variable order operators.

(4) Processes with both jump and continuous parts.